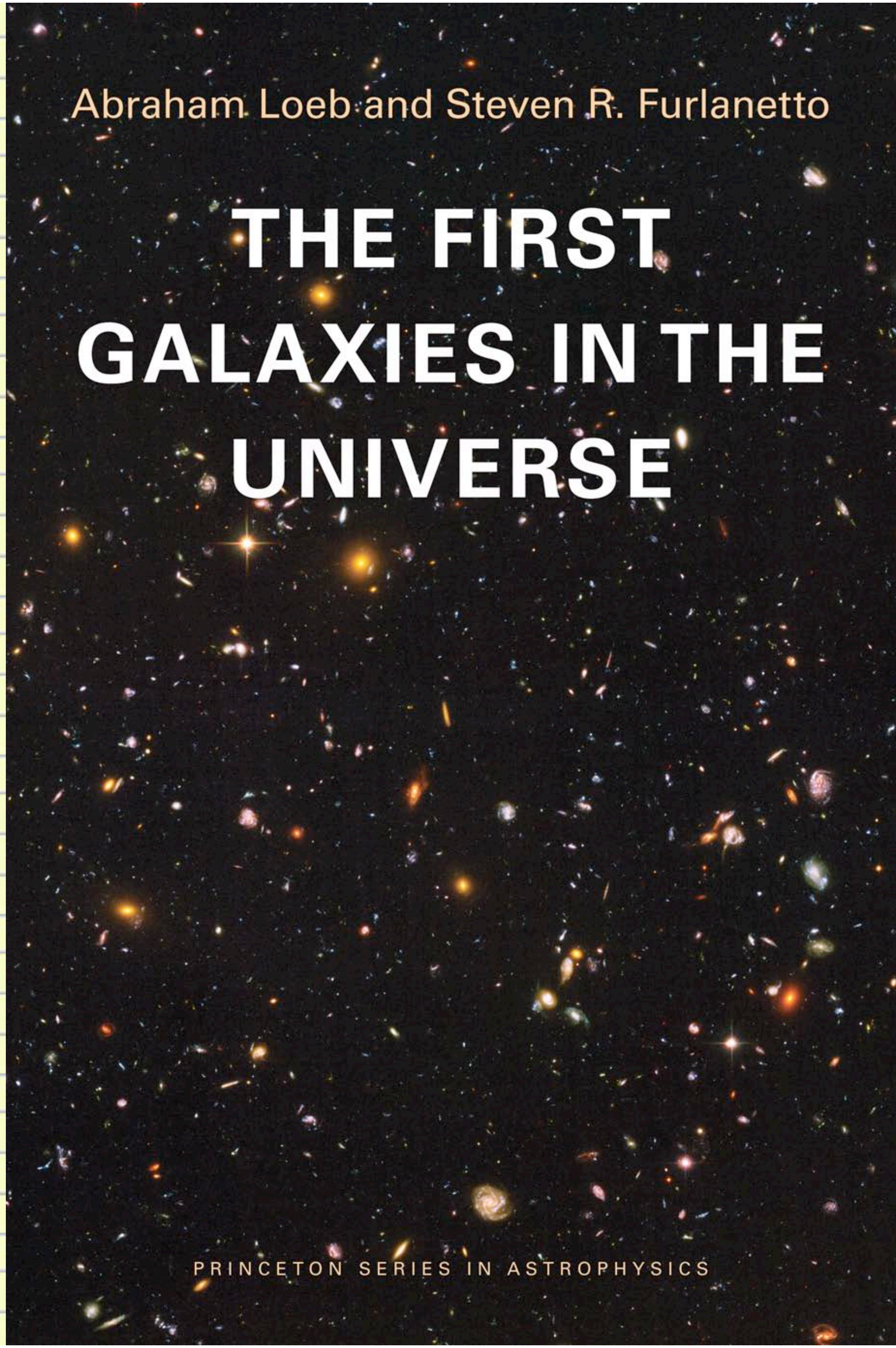
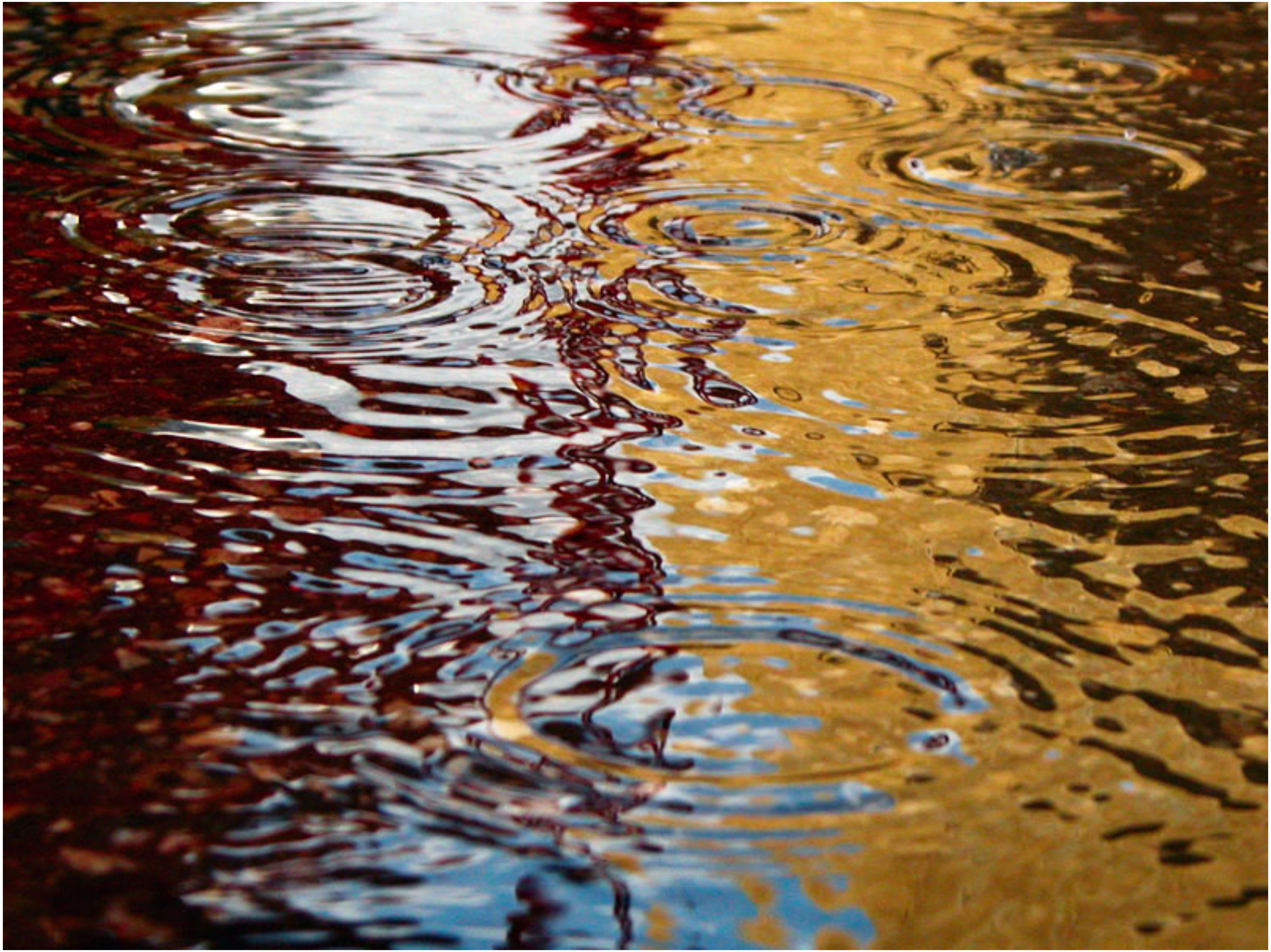
Lecture I: *Structure Formation in the Universe*

Avi Loeb
Harvard University


Abraham Loeb and Steven R. Furlanetto
THE FIRST GALAXIES IN THE UNIVERSE
PRINCETON SERIES IN ASTROPHYSICS

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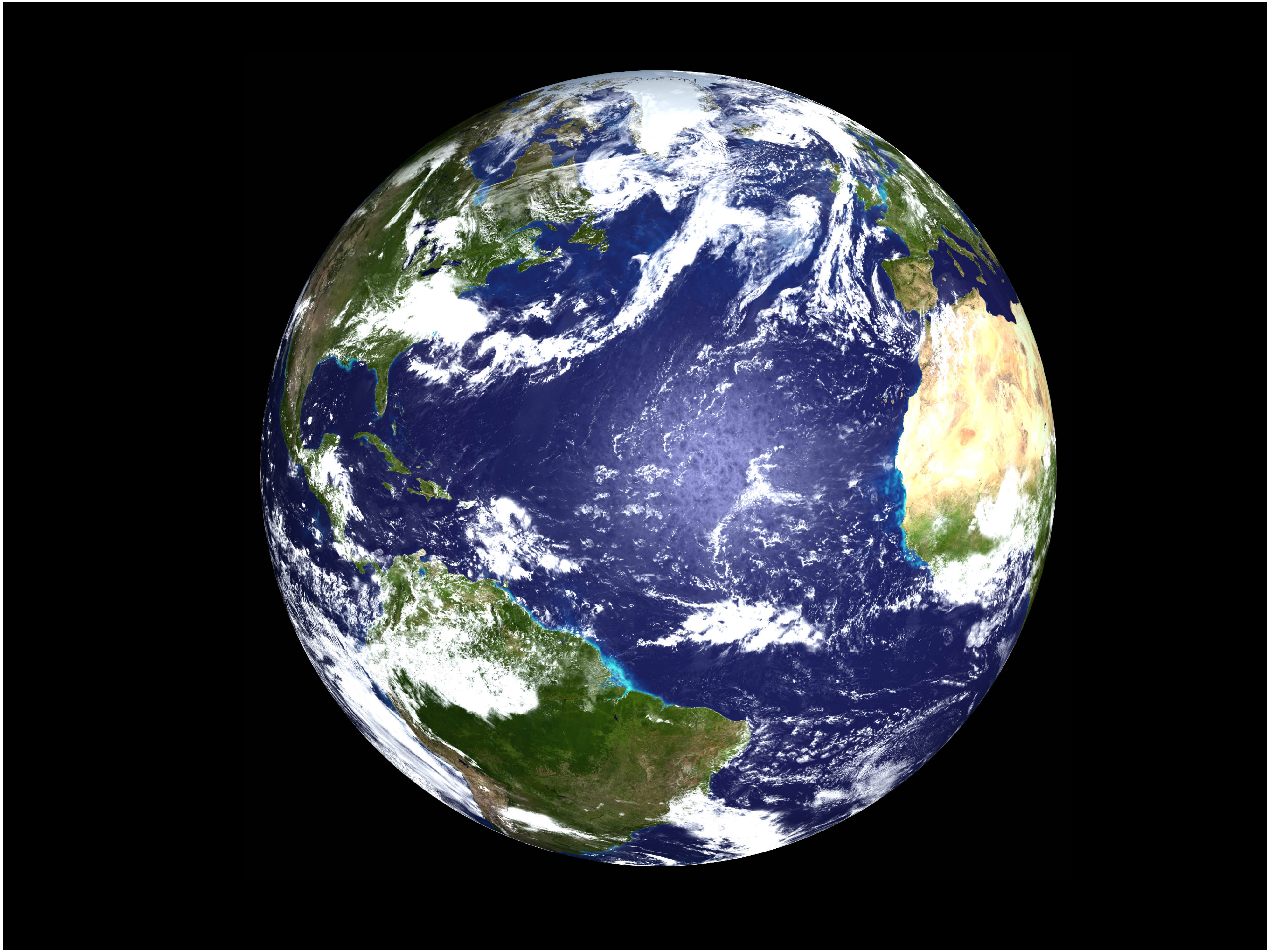



we are here

Cradle Mountain Lodge
Tasmania, January 2008

WMAP Cosmological Parameters Model: lcdm Data: all		
$10^2\Omega_b h^2$	=	$2.19^{+0.06}_{-0.08}$
A	=	$0.67^{+0.04}_{-0.05}$
$A_{0.002}$	=	$0.81^{+0.04}_{-0.05}$
$\Delta^2_{\mathcal{R}}$	=	$(20 \times 10^{-10} \pm 1 \times 10^{-10}) \times 10^{-10}$
$\Delta^2_{\mathcal{R}}(k = 0.002/Mpc)$	=	$(24 \times 10^{-10}{}^{+1\times 10^{-10}}_{-2\times 10^{-10}}) \times 10^{-10}$
h	=	$0.71^{+0.01}_{-0.02}$
H_0	=	71^{+1}_{-2} km/s/Mpc
ℓ_A	=	$303.0^{+0.9}_{-1.3}$
n_s	=	$0.938^{+0.013}_{-0.018}$
$n_s(0.002)$	=	$0.938^{+0.012}_{-0.023}$
Ω_b	=	$0.044^{+0.002}_{-0.003}$
$\Omega_b h^2$	=	$0.0220^{+0.0006}_{-0.0008}$
Ω_c	=	$0.22^{+0.01}_{-0.02}$
Ω_Λ	=	0.74 ± 0.02
Ω_m	=	$0.26^{+0.01}_{-0.03}$
$\Omega_m h^2$	=	$0.131^{+0.004}_{-0.010}$
r_s	=	148^{+1}_{-2} Mpc
b_{SDSS}	=	$0.95^{+0.05}_{-0.06}$
σ_8	=	$0.75^{+0.03}_{-0.04}$
$\sigma_8\Omega_m^{0.6}$	=	$0.34^{+0.02}_{-0.03}$
A_{SZ}	=	$0.78^{+0.23}_{-0.78}$
t_0	=	$13.8^{+0.1}_{-0.2}$ Gyr
τ	=	$0.069^{+0.026}_{-0.029}$
θ_A	=	$0.594 \pm 0.002\ °$
z_{eq}	=	3135^{+85}_{-159}
z_r	=	$9.3^{+2.8}_{-2.0}$

The initial conditions of the Universe can be summarized on a single sheet of paper, yet thousands of books cannot fully describe the complex structures we see today… Why?

Gravitational instability

Standard Cosmological Model

- On large scales: homogeneous and isotropic

$$ds^2 = c^2 dt^2 - d\ell^2$$

$$d\ell^2 = a(t)^2(dx^2 + dy^2 + dz^2) = a^2(t)(dr^2 + r^2 d\Omega)$$

- Hubble expansion

A source located at a separation $R = a(t)r$

$$v = dR/dt = \dot{a}r = (\dot{a}/a)R$$

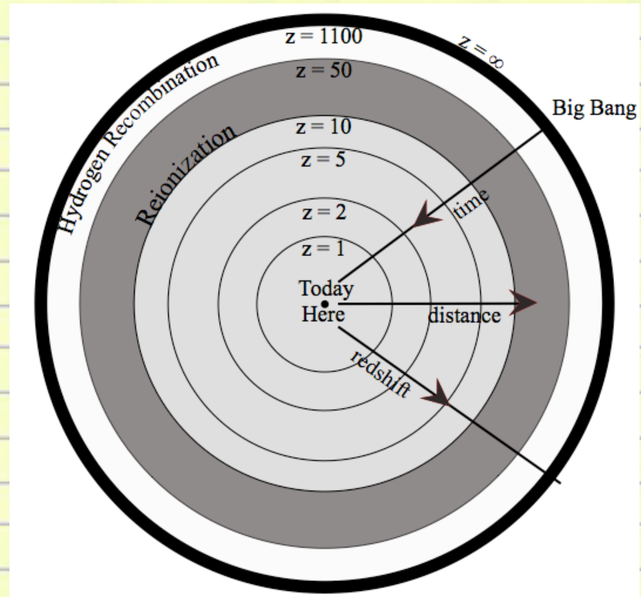
$$v = HR \qquad H = \dot{a}/a$$

$$\frac{\Delta\nu}{\nu} \approx -\frac{\Delta v}{c} = -\left(\frac{\dot{a}}{a}\right)\left(\frac{R}{c}\right) = -\frac{(\dot{a}\Delta t)}{a} = -\frac{\Delta a}{a}$$

$$\nu \propto a^{-1}$$

$$\lambda = (c/\nu) \propto a \qquad a = 1/(1+z)$$

$$\lambda_{\rm dB} = (h/p) \propto a$$



Expansion History

- Gravitating mass density $\rho_{\rm grav} = (\rho + 3p/c^2)$

$$p_{\rm rad}/c^2 = \tfrac{1}{3}\rho_{\rm rad} \qquad \rho_{\rm grav} = 2\rho_{\rm rad}$$

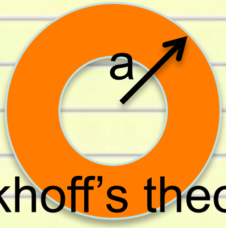
$$p_{\rm vac}/c^2 = -\rho_{\rm vac} \qquad \rho_{\rm grav} = (\rho_{\rm vac} + 3p_{\rm vac}/c^2) = -2\rho_{\rm vac}$$

$$\rho_{\rm matter} \propto a^{-3}$$

$$\rho_{\rm rad}c^2 \propto a^{-4}$$

$$\rho_{\rm vac}c^2\Delta V = \Delta E_{\rm vac} = -p_{\rm vac}\Delta V$$

- Acceleration



Birkhoff's theorem

$$\frac{d^2a}{dt^2} = -\frac{GM_{\rm grav}}{a^2} \qquad M_{\rm grav} = \rho_{\rm grav}V$$

$$V = \tfrac{4\pi}{3}a^3$$

$$d(\rho c^2 V) = -pdV \qquad -3pa\dot{a}/c^2 = a^2\dot{\rho} + 3\rho a\dot{a}$$

$$E = \frac{1}{2}\dot{a}^2 - \frac{GM}{a} \qquad M = \rho V$$

Expansion History

$$\frac{E}{\dot{a}^2/2} = 1 - \Omega,$$

where $\Omega = \rho/\rho_c$, with

$$\rho_c = \frac{3H^2}{8\pi G} = 9.2 \times 10^{-30} \frac{\mathrm{g}}{\mathrm{cm}^3} \left(\frac{H}{70\ \mathrm{km\ s^{-1} Mpc^{-1}}} \right)^2$$

- Hubble expansion rate

a flat universe with $E = 0$ satisfies

$$\frac{H(t)}{H_0} = \left[\frac{\Omega_m}{a^3} + \Omega_\Lambda + \frac{\Omega_r}{a^4} \right]^{1/2}$$

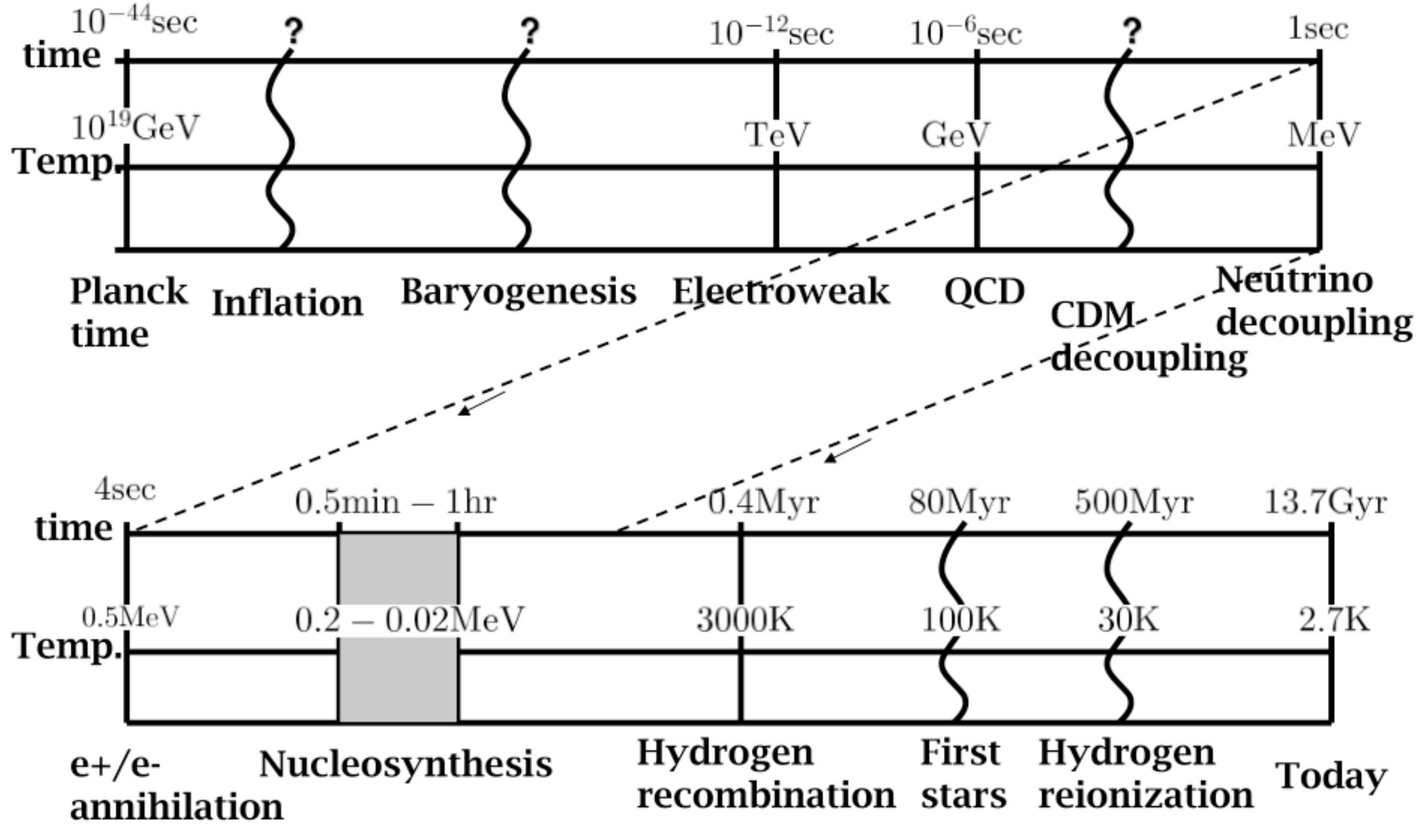
where we define H_0 and $\Omega_0 = (\Omega_m + \Omega_\Lambda + \Omega_r) = 1$ to be the present-day values of H and Ω, respectively.

Vacuum domination: $H \equiv (\dot{a}/a) = const \rightarrow a \propto \exp\{Ht\}$

- Age of the Universe at redshift 1<z<1000

$$t \approx \frac{2}{3H_0{\Omega_m}^{1/2}(1+z)^{3/2}} = \frac{0.95\times10^9\ \text{years}}{[(1+z)/7]^{3/2}}$$

Thermal History



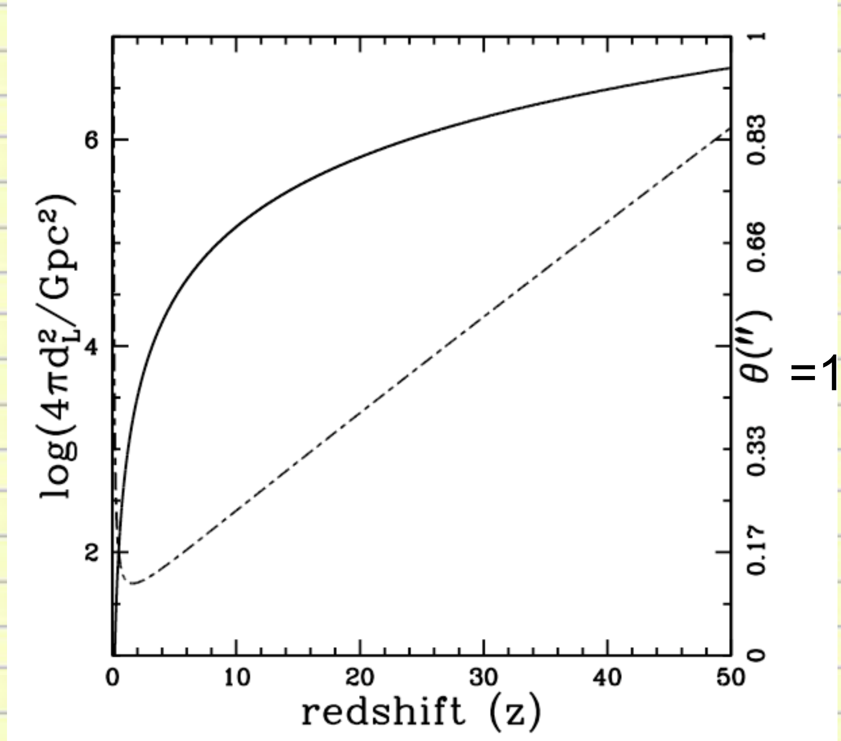
Distances

- Luminosity distance: observed flux

$$f = \frac{L}{4\pi d_{\mathrm{L}}^2}$$

$$f = \frac{L dt_{\mathrm{em}}/(1+z)}{4\pi r_{\mathrm{em}}^2 dt_{\mathrm{obs}}} = \frac{L}{4\pi r_{\mathrm{em}}^2 (1+z)^2}$$

$$d_{\mathrm{L}} = r_{\mathrm{em}}(1+z) = d_{\mathrm{A}}(1+z)^2$$

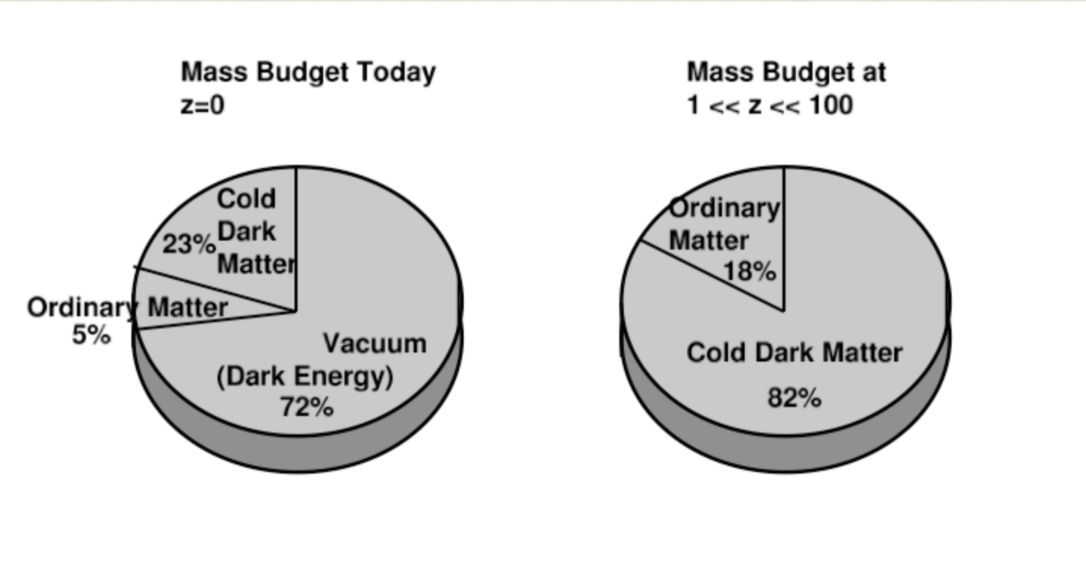


=1kpc

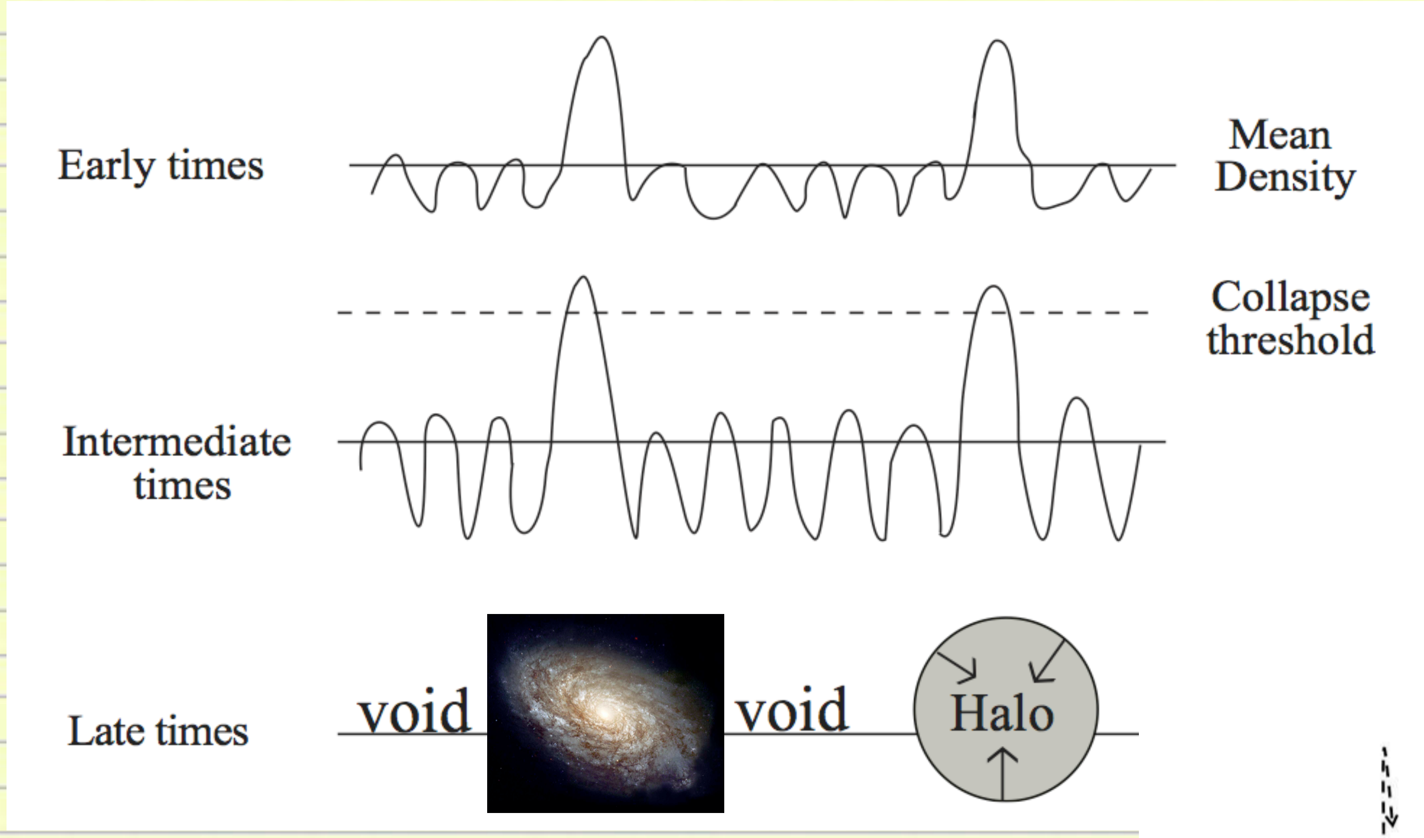
$$r_{\mathrm{em}} = \int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{c\,dt}{a(t)} = \frac{c}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_m (1+z')^3 + \Omega_\Lambda}}$$

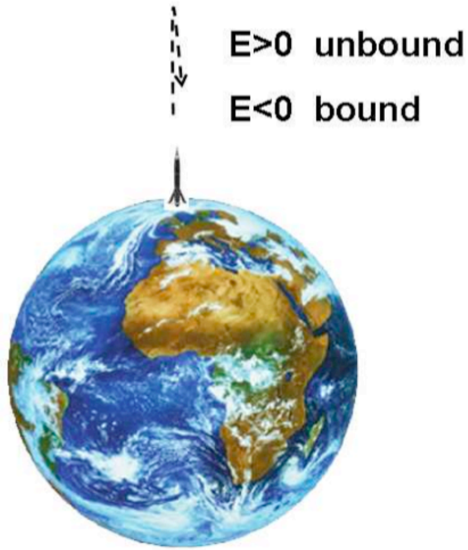
Parameters of the Standard Cosmological Model

Ω_Λ	Ω_m	Ω_b	h	n_s	σ_8
0.72	0.28	0.05	0.7	1	0.82



Growth of Density Perturbations





$$\delta(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\bar{\rho}} - 1$$

deviation from the Hubble flow $\mathbf{u} \equiv \mathbf{v} - H\mathbf{r}$

Linear (small) Perturbations

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot [(1+\delta)\mathbf{u}] = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + H\mathbf{u} + \frac{1}{a}(\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{a}\nabla\phi - \frac{1}{a\bar{\rho}}\nabla(\delta p)$$

$$\nabla^2\phi = 4\pi G\bar{\rho}a^2\delta$$

Pressure: zero for cold dark matter; finite for gas

$$\frac{\partial^2\delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} = 4\pi G\bar{\rho}\delta - \frac{c_s^2 k^2}{a^2}\delta$$

Growth factor during matter domination (without pressure):

$$D(t) \propto \frac{(\Omega_\Lambda a^3 + \Omega_m)^{1/2}}{a^{3/2}} \int_0^a \frac{a'^{3/2}\, da'}{(\Omega_\Lambda a'^3 + \Omega_m)^{3/2}} \quad \propto a \quad (z \gg 1)$$

But only logarithmic growth during radiation domination ($c_s \sim c/\sqrt{3}$)

Fourier Space

$$\delta_{\mathbf{k}} = \int d^3x\, \delta(x) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\delta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}$$

- Power spectrum: $P(\mathbf{k}) = \langle \delta_{\mathbf{k}} \delta^*_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{k}') P(\mathbf{k})$

$$(\delta M/M)^2 \propto k^3 P(k) \text{ for } k \sim 2\pi/\ell$$

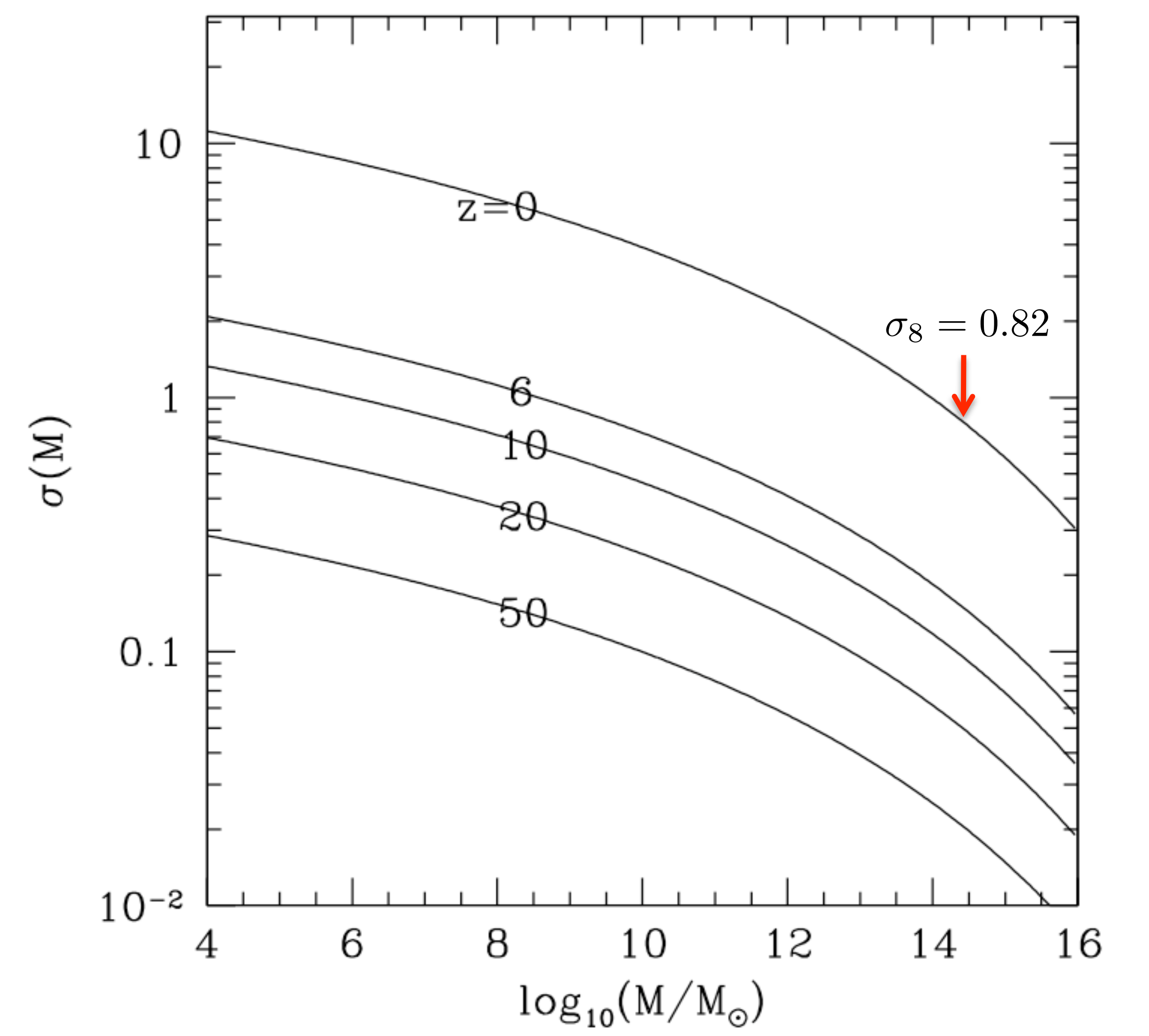
primordial power-law spectrum $P(k) \propto k^{n_s}$ with $n_s \approx 1$

With cold dark matter, turnover at matter-radiation equality: $P(k) \propto k^{n_s - 4}$

the gravitational potential, $\sim (G\delta M/\ell) \propto \ell^{(1-n_s)/2}$, is independent of scale if $n_s = 1$

(as expected from quantum fluctuations generated during a period of inflation with *H=const* and all modes exiting the horizon with the same amplitude)

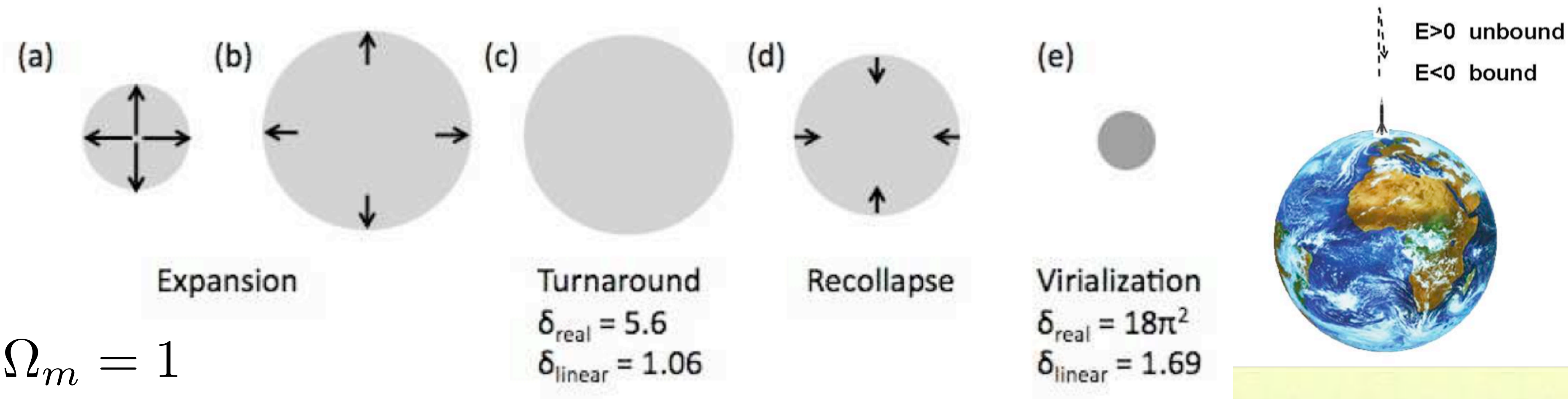
Amplitude of Density Perturbations



$$p(\delta)d\delta = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\delta^2/2\sigma^2} d\delta$$

Nonlinear Spherical Collapse

$$\frac{d^2 r}{dt^2} = H_0^2 \Omega_\Lambda \, r - \frac{GM}{r^2}$$



$$\Omega_m = 1$$

Stages of the spherical collapse model. At first, the overdensity (in gray) expands, though its excess gravity quickly slows that expansion below that of the Hubble flow. When the real fractional overdensity reaches $9\pi^2/16 \approx 5.6$ (corresponding to a *linearized* overdensity of 1.06), the expansion stops at *turnaround* and then begins to recollapse. When the overdensity reaches $18\pi^2 \approx 178$ (corresponding to a *linearized* overdensity of 1.69), the perturbation *virializes* as a collapsed dark matter halo.

Properties of Dark Matter Halos

- Density contrast at virialization

$$\Delta_{\rm vir}(\Omega_m = 1) \equiv \frac{\rho_{\rm vir}(z_{\rm vir})}{\bar{\rho}_{\rm crit}(z_{\rm vir})} = \left(\frac{9\pi^2}{16}\right) \times 8 \times 4 = 18\pi^2 \approx 178$$

- Halo radius

$$r_{\rm vir} = 1.5 \left[\frac{\Omega_m}{\Omega_m(z)} \frac{\Delta_c}{18\pi^2}\right]^{-1/3} \left(\frac{M}{10^8 M_\odot}\right)^{1/3} \left(\frac{1+z}{10}\right)^{-1} \mathrm{kpc}$$

- Circular velocity

$$V_c = \left(\frac{GM}{r_{\rm vir}}\right)^{1/2} = 17.0 \left[\frac{\Omega_m}{\Omega_m(z)} \frac{\Delta_c}{18\pi^2}\right]^{1/6} \left(\frac{M}{10^8 M_\odot}\right)^{1/3} \left(\frac{1+z}{10}\right)^{1/2} \mathrm{km\ s^{-1}}$$

- Virial temperature

$$T_{\rm vir} = \frac{\mu m_p V_c^2}{2k} = 1.04\times 10^4 \left(\frac{\mu}{0.6}\right) \left[\frac{\Omega_m}{\Omega_m(z)} \frac{\Delta_c}{18\pi^2}\right]^{1/3} \left(\frac{M}{10^8 M_\odot}\right)^{2/3} \left(\frac{1+z}{10}\right) \mathrm{K}$$

Aquarius N-body Simulation

(Springel et al. 2011)

"Universal" dark matter density profile

NFW

$$\rho(r) = \frac{3H_0^2}{8\pi G}(1+z)^3 \frac{\Omega_m}{\Omega_m(z)} \frac{\delta_c}{c_{\rm N}x(1+c_{\rm N}x)^2}$$

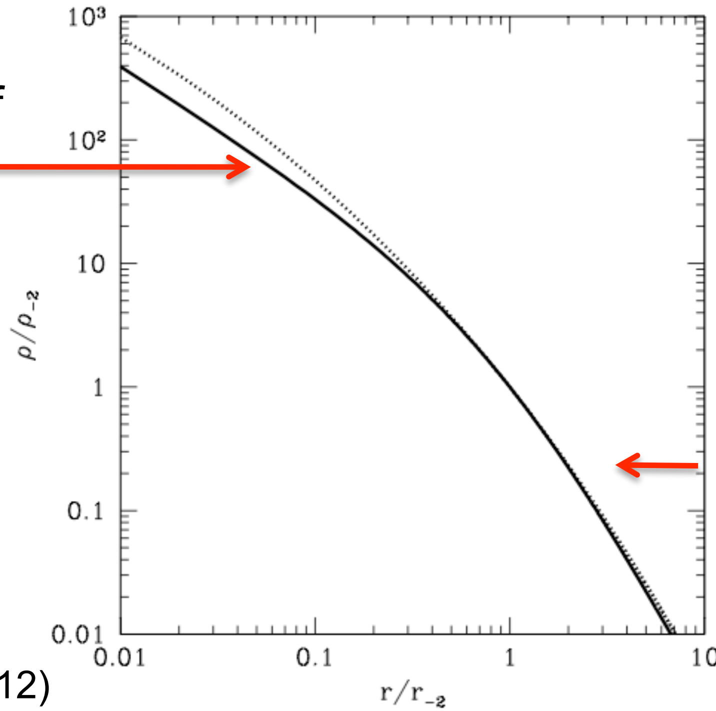
$$x = r/r_{\rm vir}$$

Einasto

$$\ln\left[\frac{\rho(r)}{\rho_{-2}}\right] = -\frac{2}{\alpha}\left[\left(\frac{r}{r_{-2}}\right)^{\alpha} - 1\right]$$

Radial orbit instability: isotropy below the half mass radius

$$\rho(r) = \frac{1}{4\pi r^2}\frac{dt}{dr} \propto r^{-(2-\epsilon)}$$



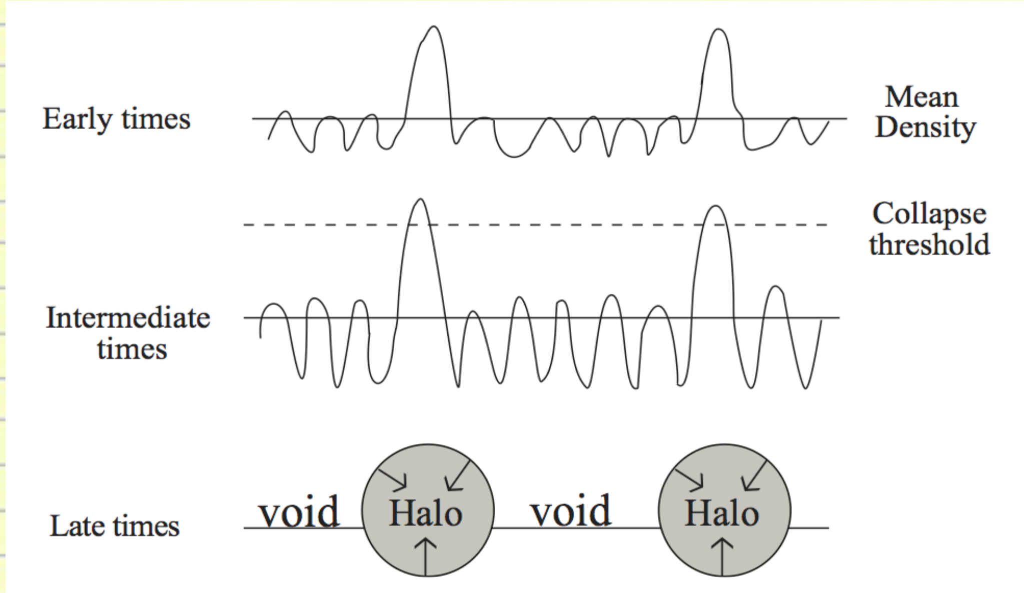
Radial infall: slope should be sensitive to environment (-4 for an isolated halo)

$$\rho(r) = \frac{N(E)}{4\pi r^2}\frac{dE}{dr} \propto r^{-4}$$

(Visbal, Loeb, & Hernquist 2012)

Abundance of Halos

- Press-Schechter Model



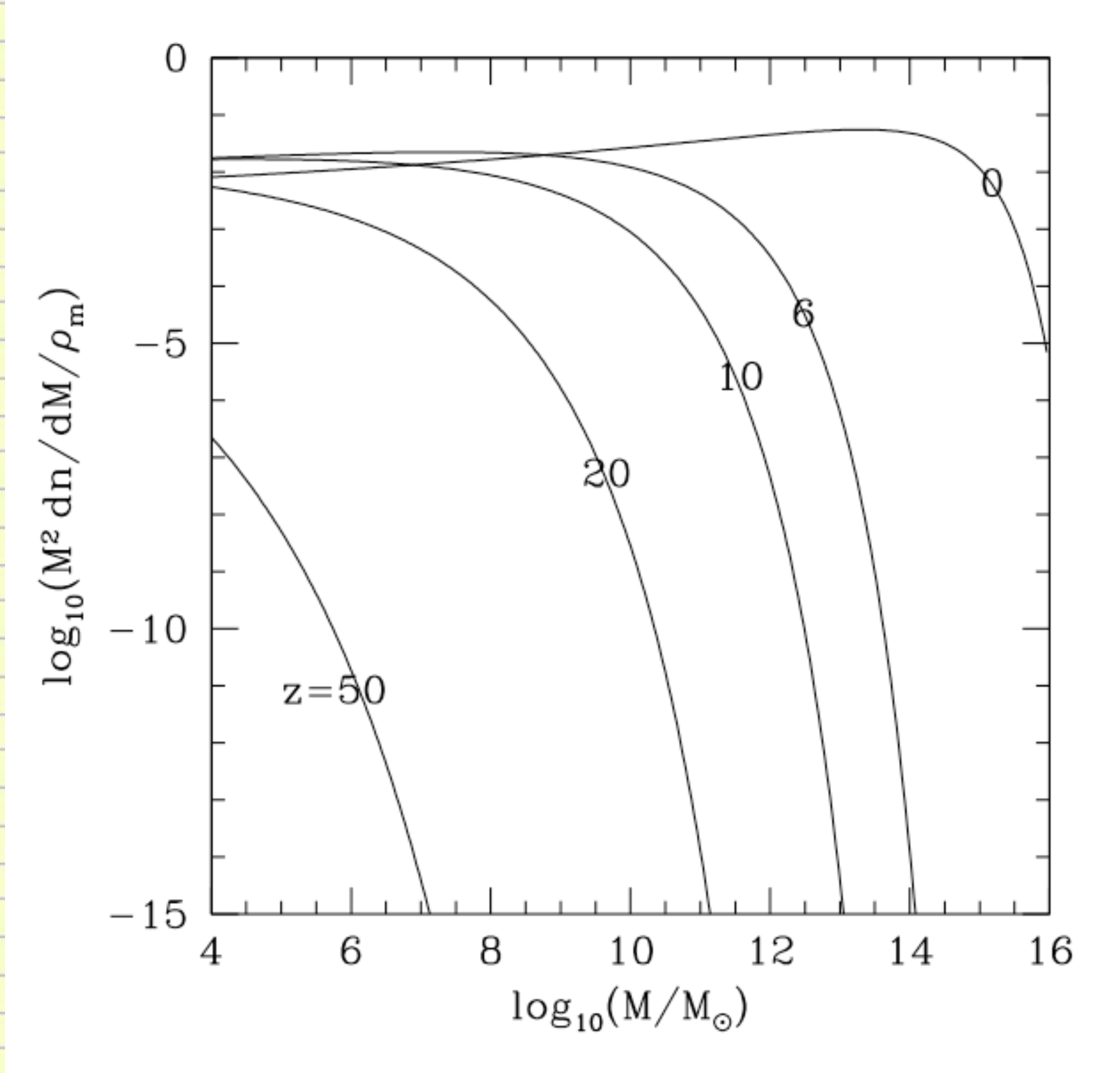
$$p(\delta)d\delta = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\delta^2/2\sigma^2} d\delta$$

$$f_{\rm coll}(>M|z) = {\rm erfc}\left(\frac{\delta_{\rm crit}(z)}{\sqrt{2}\,\sigma(M)}\right)$$

$$n(M) = \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M} \frac{-d(\ln\sigma)}{dM} \nu_c\, e^{-\nu_c^2/2}$$

$$\nu_c = \delta_{\rm crit}(z)/\sigma(M)$$

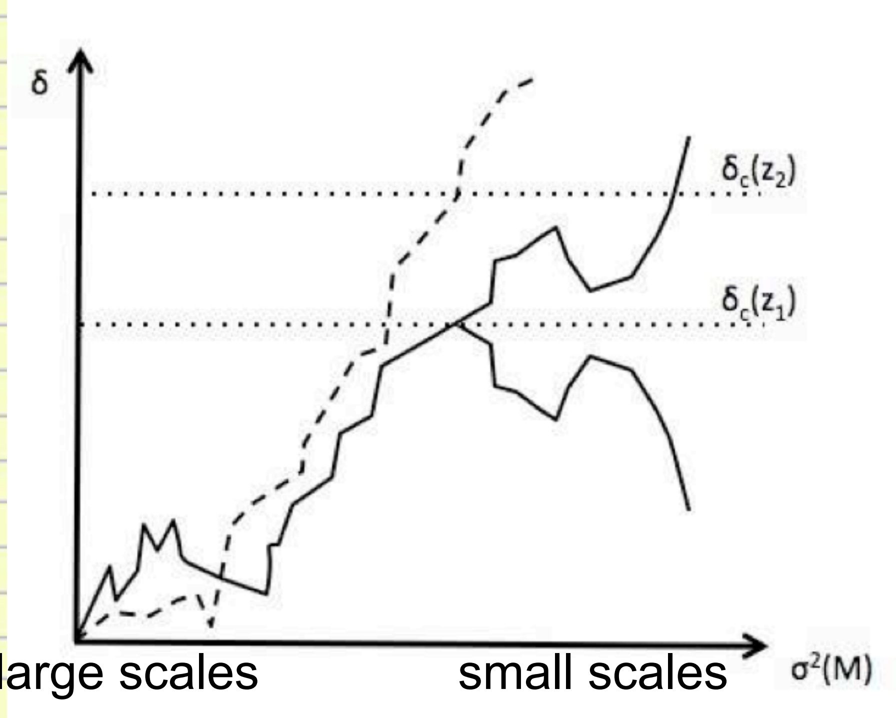
Halo mass budget



Excursion Set Formalism

$$\delta_M \propto \int_{k<k_c(M)} \frac{d^3k}{(2\pi)^3}\delta_k,$$

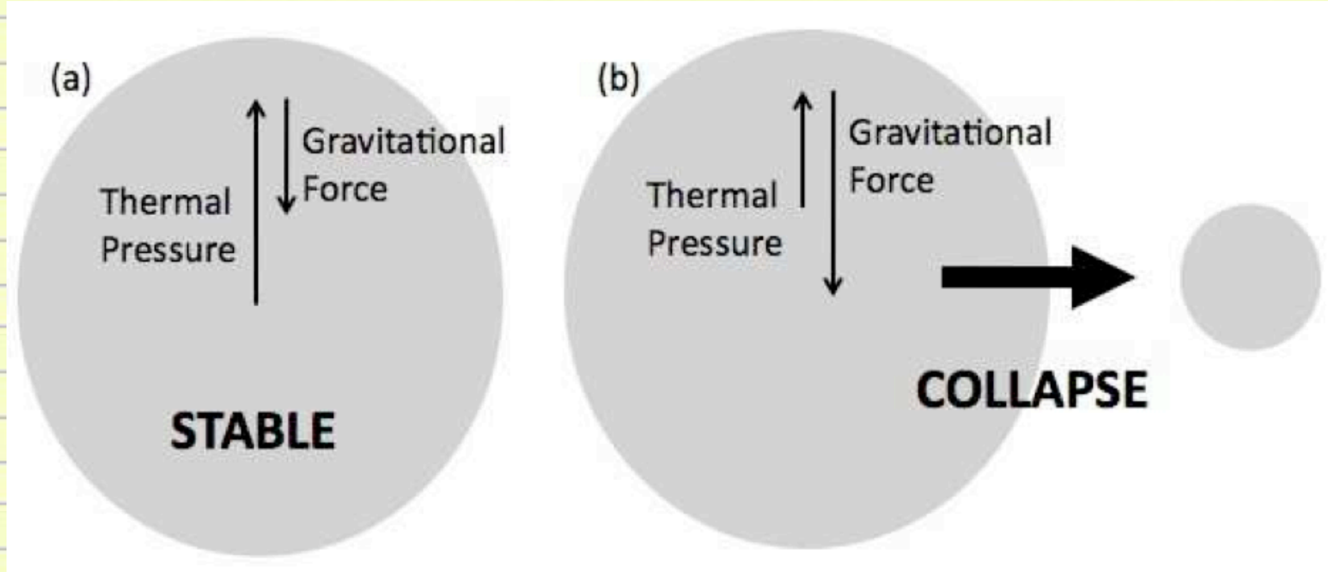
Recovers missing factor of 2 in the Press-Schechter approach



We therefore compute the distribution of halo masses M_1 that a mass element finds itself in at redshift z_1, given that it is part of a larger halo of mass M_2 at a later redshift $z_2 < z_1$:

$$\frac{dP}{dM_1}(M_1, z_1|M_2, z_2) =$$
$$\sqrt{\frac{2}{\pi}}\left[\frac{\delta_{\rm crit}(z_1)-\delta_{\rm crit}(z_2)}{\sigma^2(M_1)-\sigma^2(M_2)}\right]\left|\frac{d\sigma(M_1)}{dM_1}\right|\exp\left\{-\frac{[\delta_{\rm crit}(z_1)-\delta_{\rm crit}(z_2)]^2}{2[\sigma^2(M_1)-\sigma^2(M_2)]}\right\}.$$

Gas Pressure



- Jeans mass

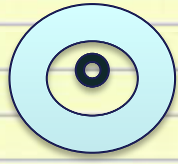
$$\frac{\lambda_J}{c_s} \sim \frac{1}{\sqrt{G\rho_m}}$$

$$M_{\rm J} = 4.54 \times 10^3 \left(\frac{\Omega_m h^2}{0.15}\right)^{-1/2} \left(\frac{\Omega_b h^2}{0.022}\right)^{-3/5} \left(\frac{1+z}{10}\right)^{3/2} M_\odot$$

Grows to ~10^9M. after intergalactic gas is photo-heated to ~10^4K

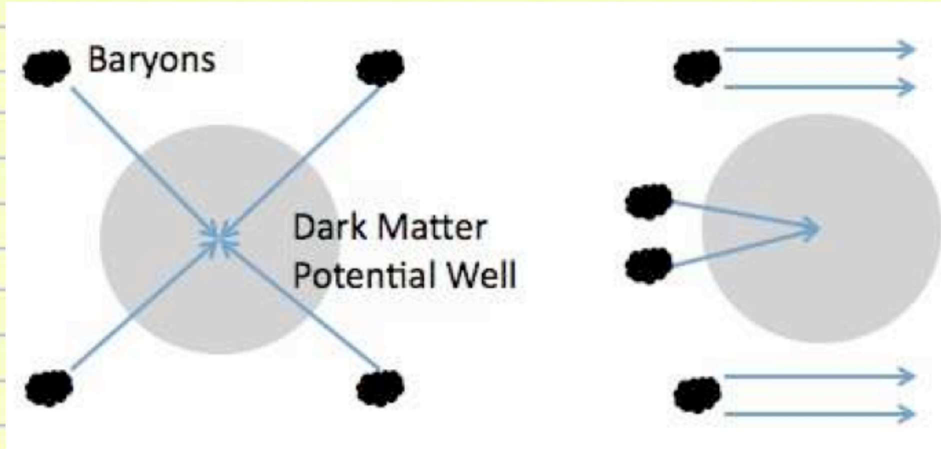
Streaming of baryons relative to Dark Matter

***Fourier transform:** acoustic oscillations (all harmonics)*



$$r = c_s t \sim \frac{ct}{\sqrt{3}} \; ; \; k_n \sim n \frac{2\pi}{r}$$

- Sound waves in the baryon-radiation fluid at z>1000, not shared by the dark matter (whose perturbations grow once matter dominates).
- Without dark matter, photon diffusion would have erased small-scale fluctuations, galaxies would not form, and we would never exist.
- Characteristic speed of “baryonic wind”: (30 km/s)x[(1+z)/1000], coherent in cells of a few cMpc.
- Affects the assembly of gas into the lowest-mass halos with M<10^6M. at 10<z<50.

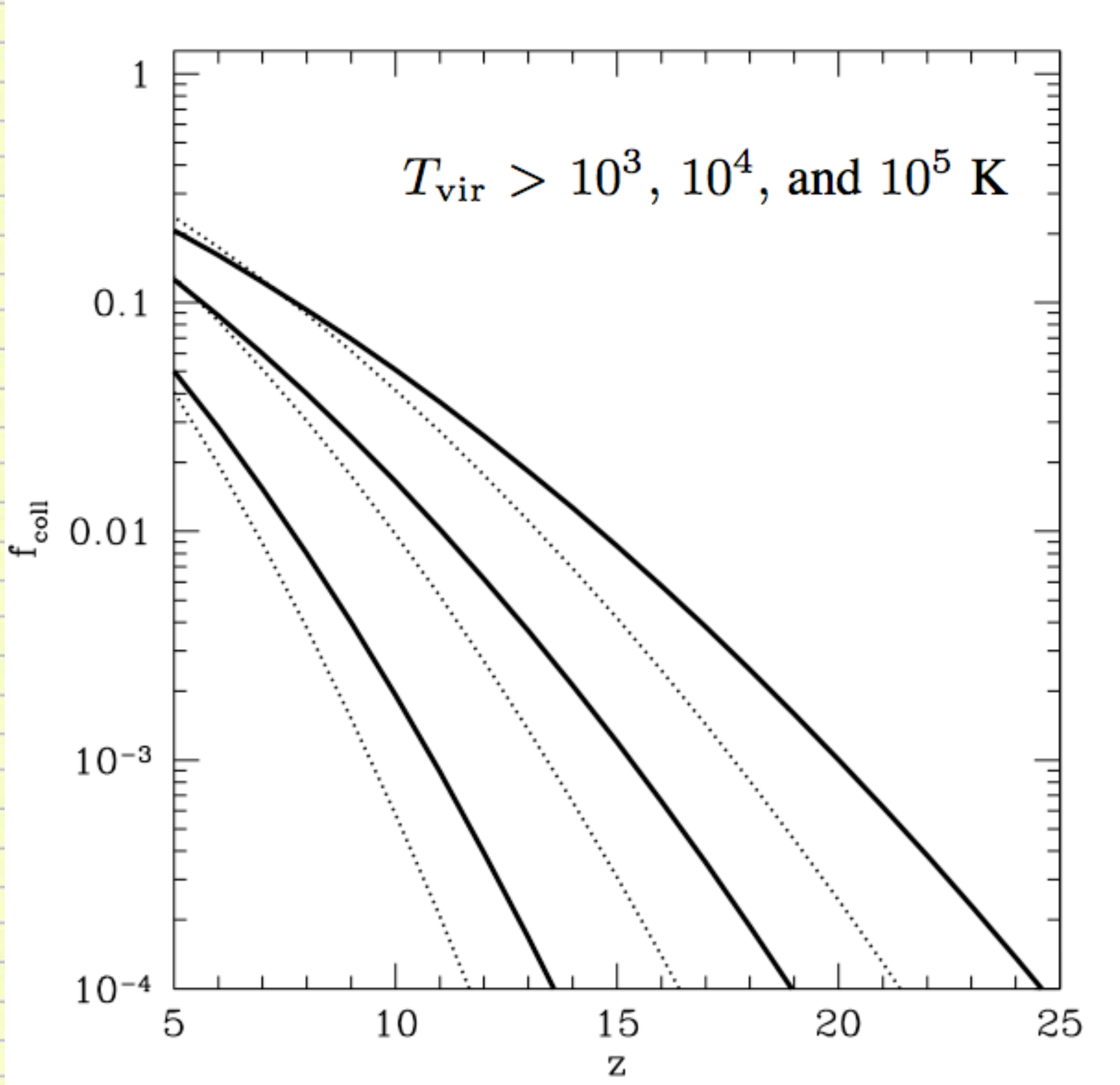


(Tseliakhovich & Hirata 2011)

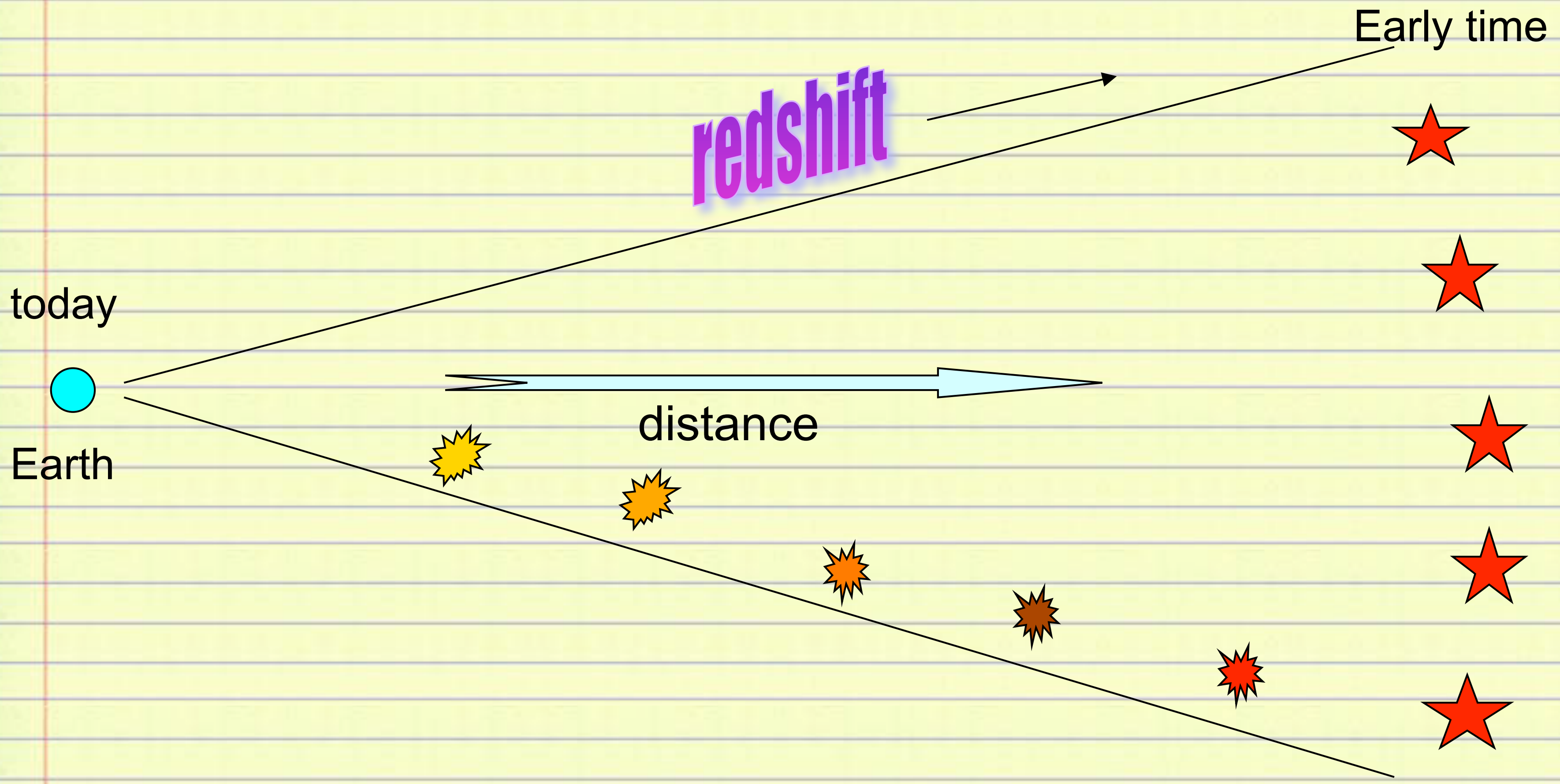
Hydrodynamic Simulation

(AREPO, Vogelsberger et al. 2011)

Fraction of collapsed matter



Cosmic-Archeology

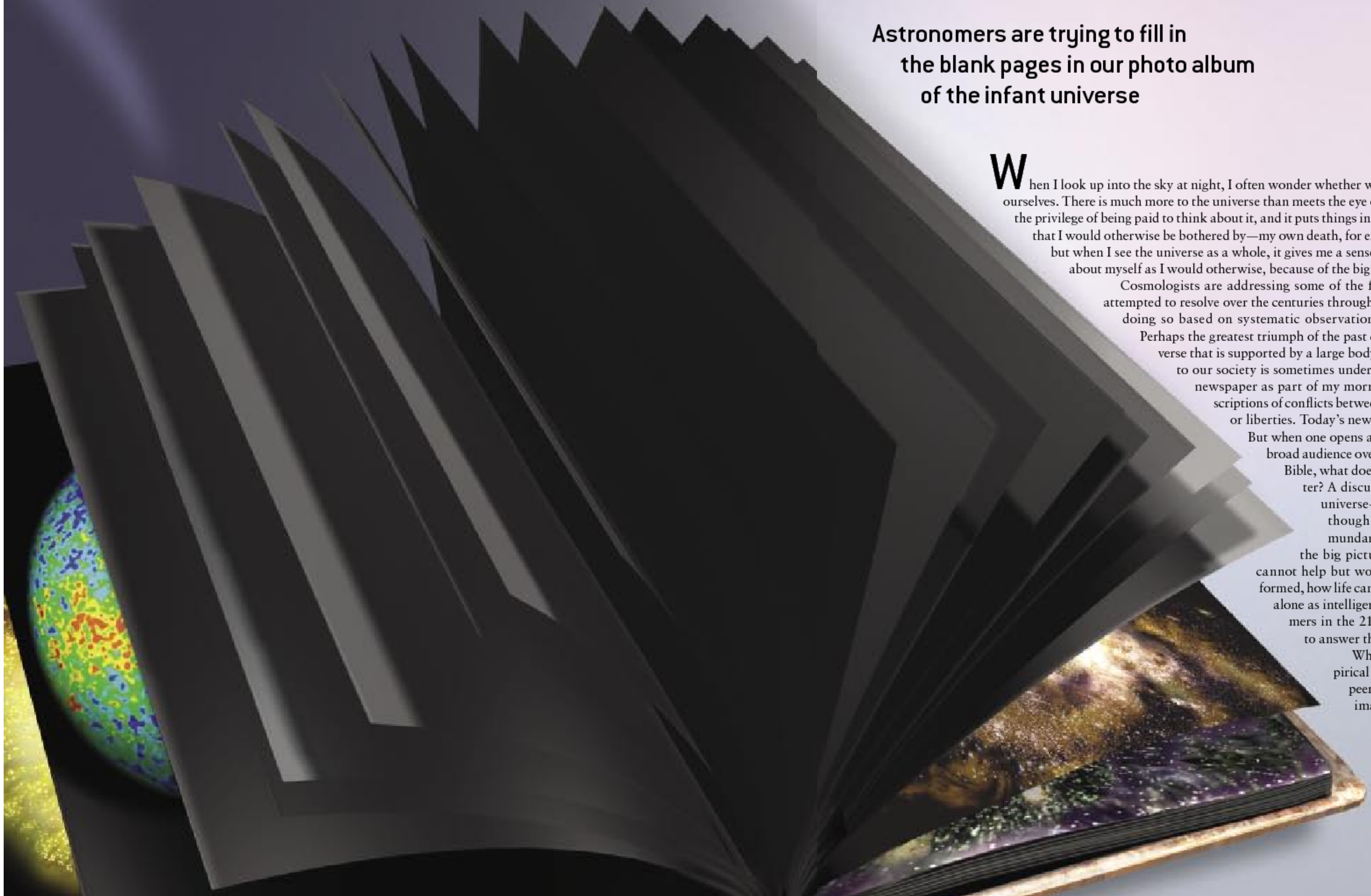


The more distant a source is, the more time it takes for its light to reach us. Hence, the light must have been emitted when the universe was younger. By looking at distant sources we can trace the history of the universe.

THE DARK AGES of the Universe

Astronomers are trying to fill in the blank pages in our photo album of the infant universe

By Abraham Loeb

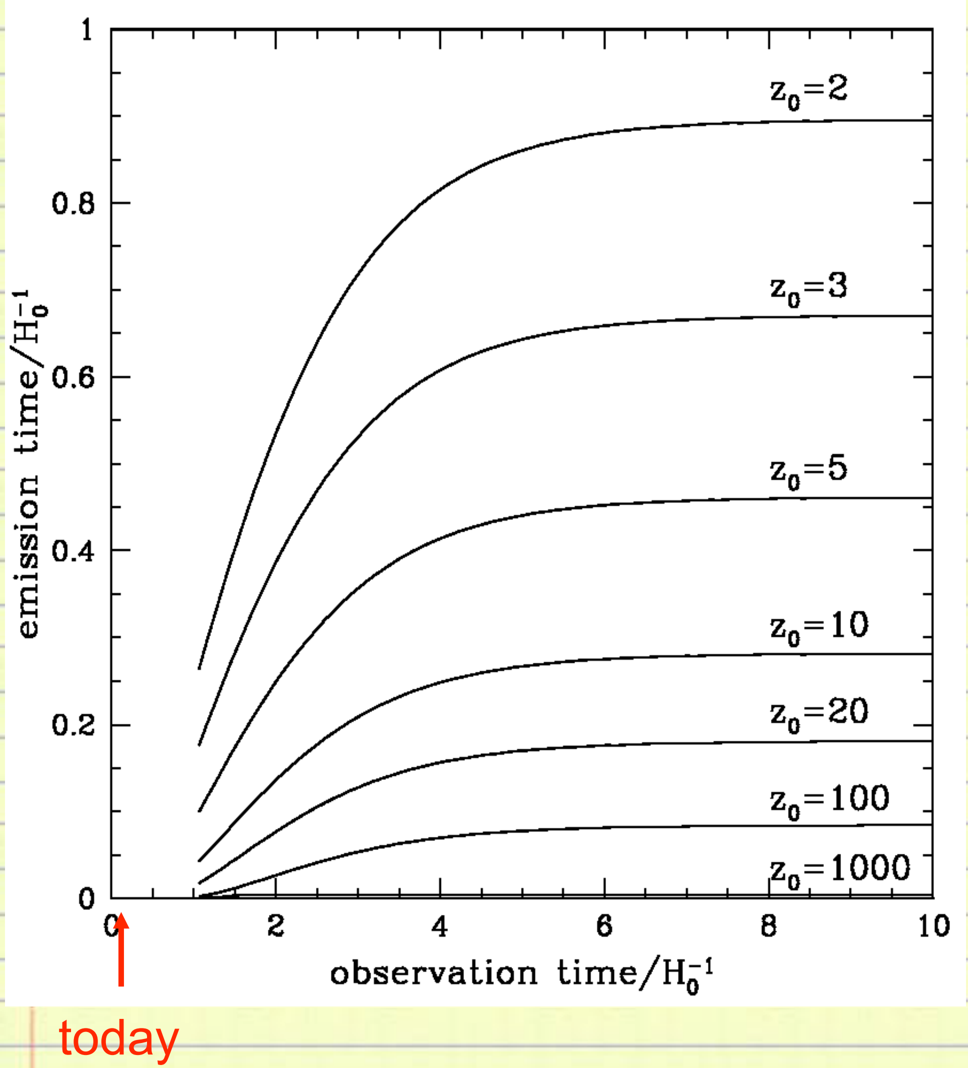


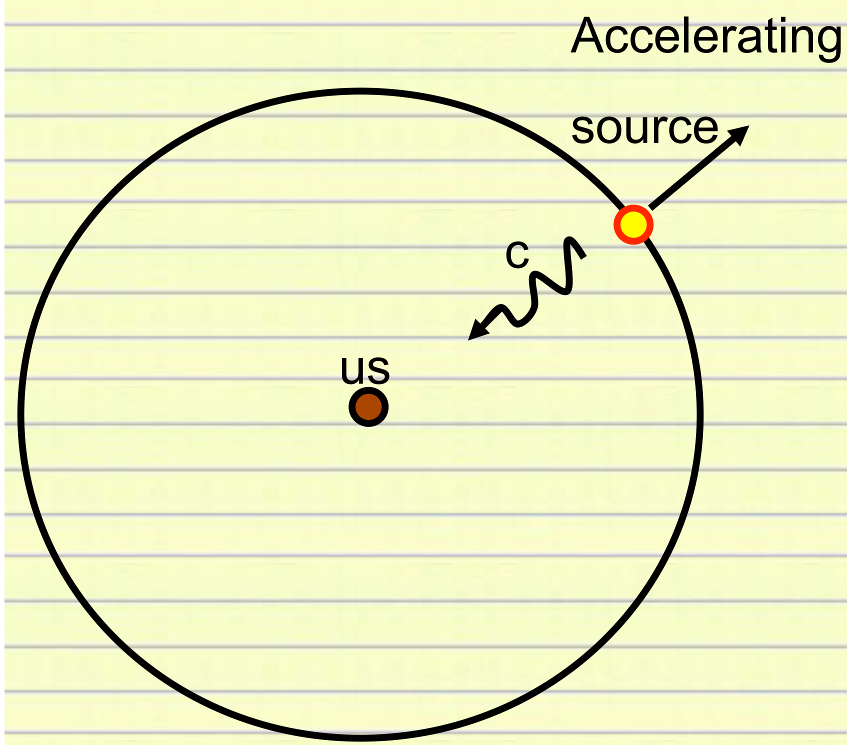
When I look up into the sky at night, I often wonder whether we humans are too preoccupied with ourselves. There is much more to the universe than meets the eye on earth. As an astrophysicist I have the privilege of being paid to think about it, and it puts things in perspective for me. There are things that I would otherwise be bothered by—my own death, for example. Everyone will die sometime, but when I see the universe as a whole, it gives me a sense of longevity. I do not care so much about myself as I would otherwise, because of the big picture.

Cosmologists are addressing some of the fundamental questions that people attempted to resolve over the centuries through philosophical thinking, but we are doing so based on systematic observation and a quantitative methodology. Perhaps the greatest triumph of the past century has been a model of the universe that is supported by a large body of data. The value of such a model to our society is sometimes underappreciated. When I open the daily newspaper as part of my morning routine, I often see lengthy descriptions of conflicts between people about borders, possessions or liberties. Today's news is often forgotten a few days later. But when one opens ancient texts that have appealed to a broad audience over a longer period of time, such as the Bible, what does one often find in the opening chapter? A discussion of how the constituents of the universe—light, stars, life—were created. Although humans are often caught up with mundane problems, they are curious about the big picture. As citizens of the universe we cannot help but wonder how the first sources of light formed, how life came into existence and whether we are alone as intelligent beings in this vast space. Astronomers in the 21st century are uniquely positioned to answer these big questions.

What makes modern cosmology an empirical science is that we are literally able to peer into the past. When you look at your image reflected off a mirror one meter

The Long Term Future of Extragalactic Astronomy



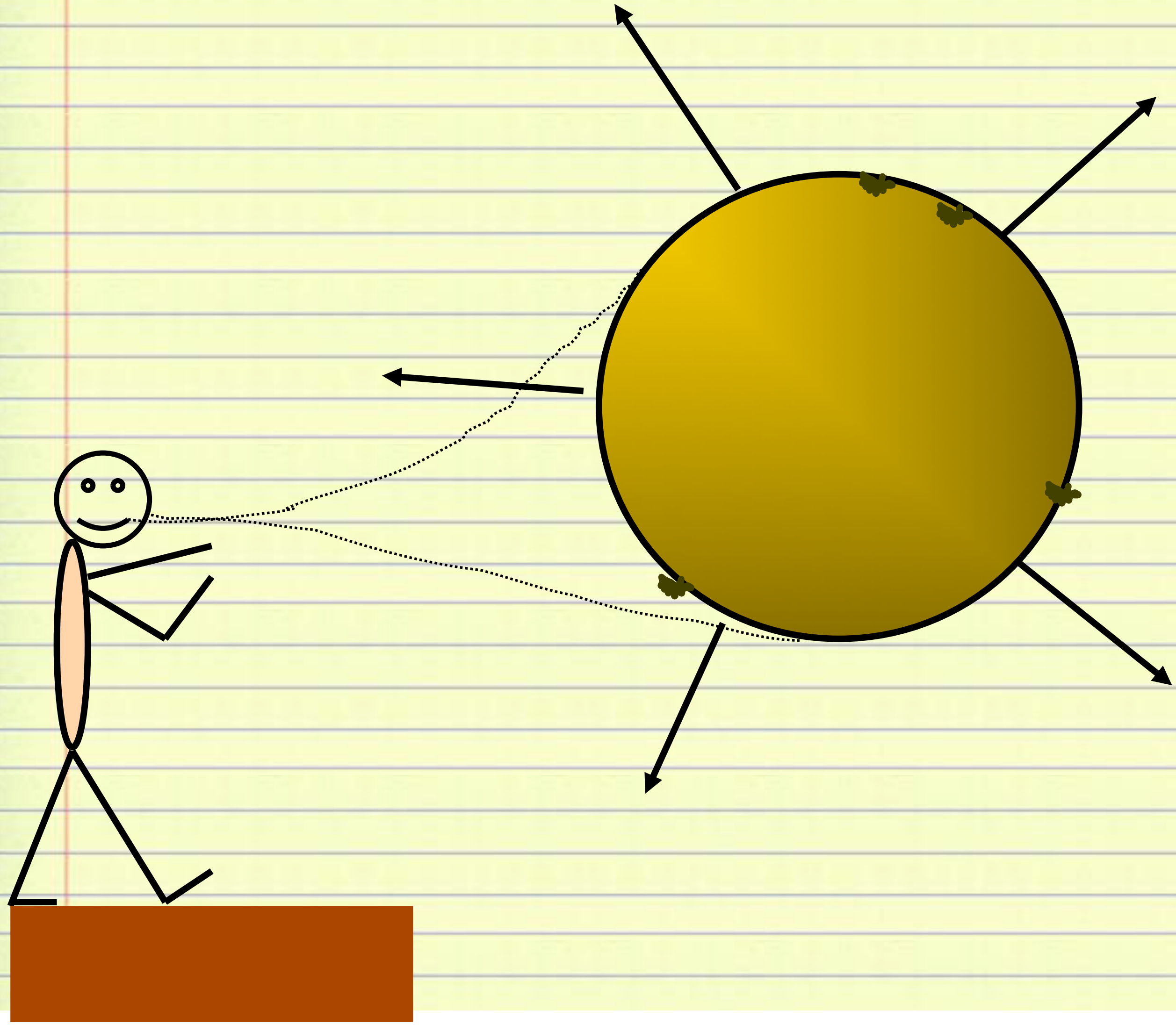


All galaxies beyond a redshift of z=1.8 are already outside our horizon (no cell phone communication to z>1.8!). (Loeb 2001)

Analogy

Ants = Photons

Balloon=Expanding Space



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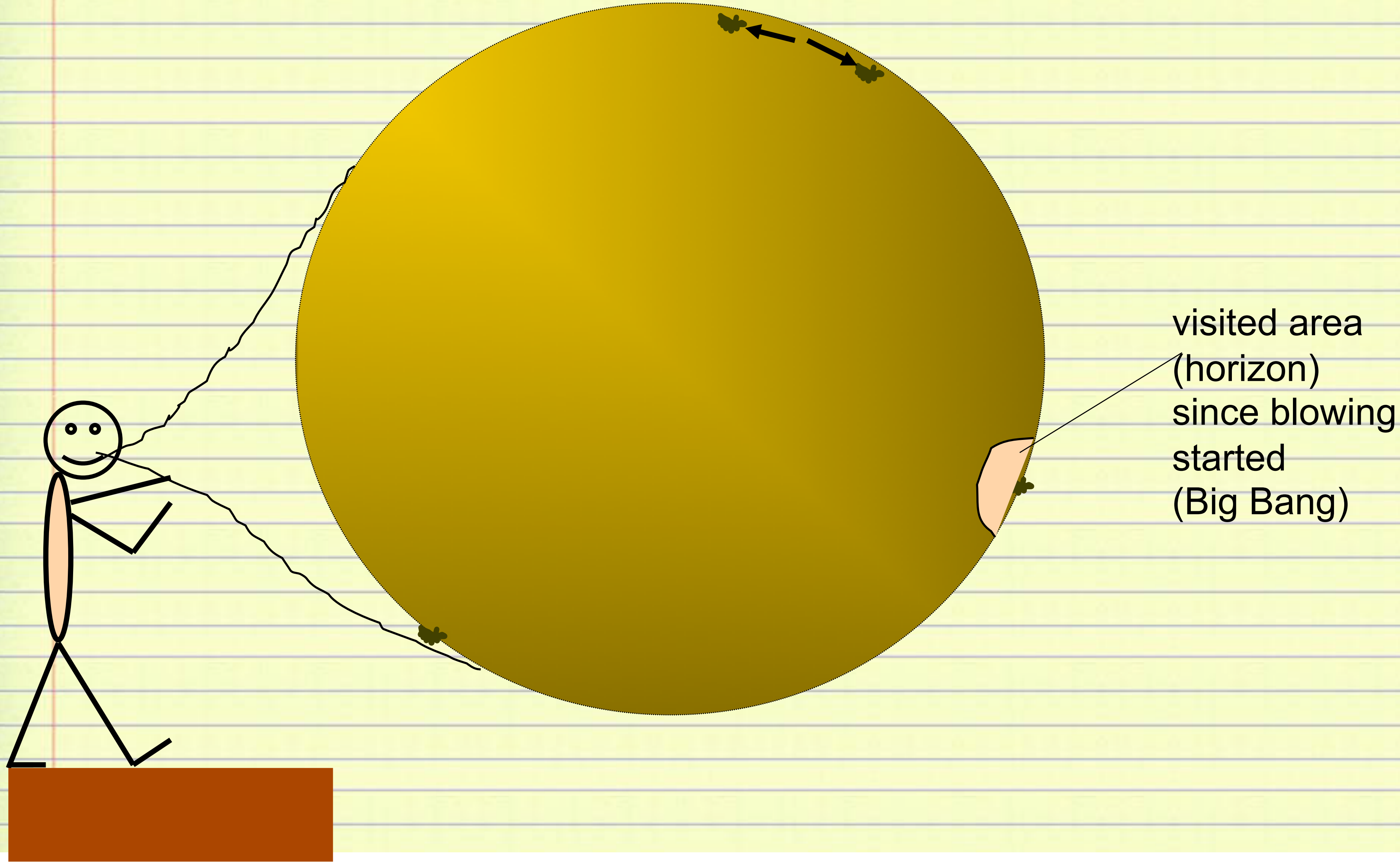
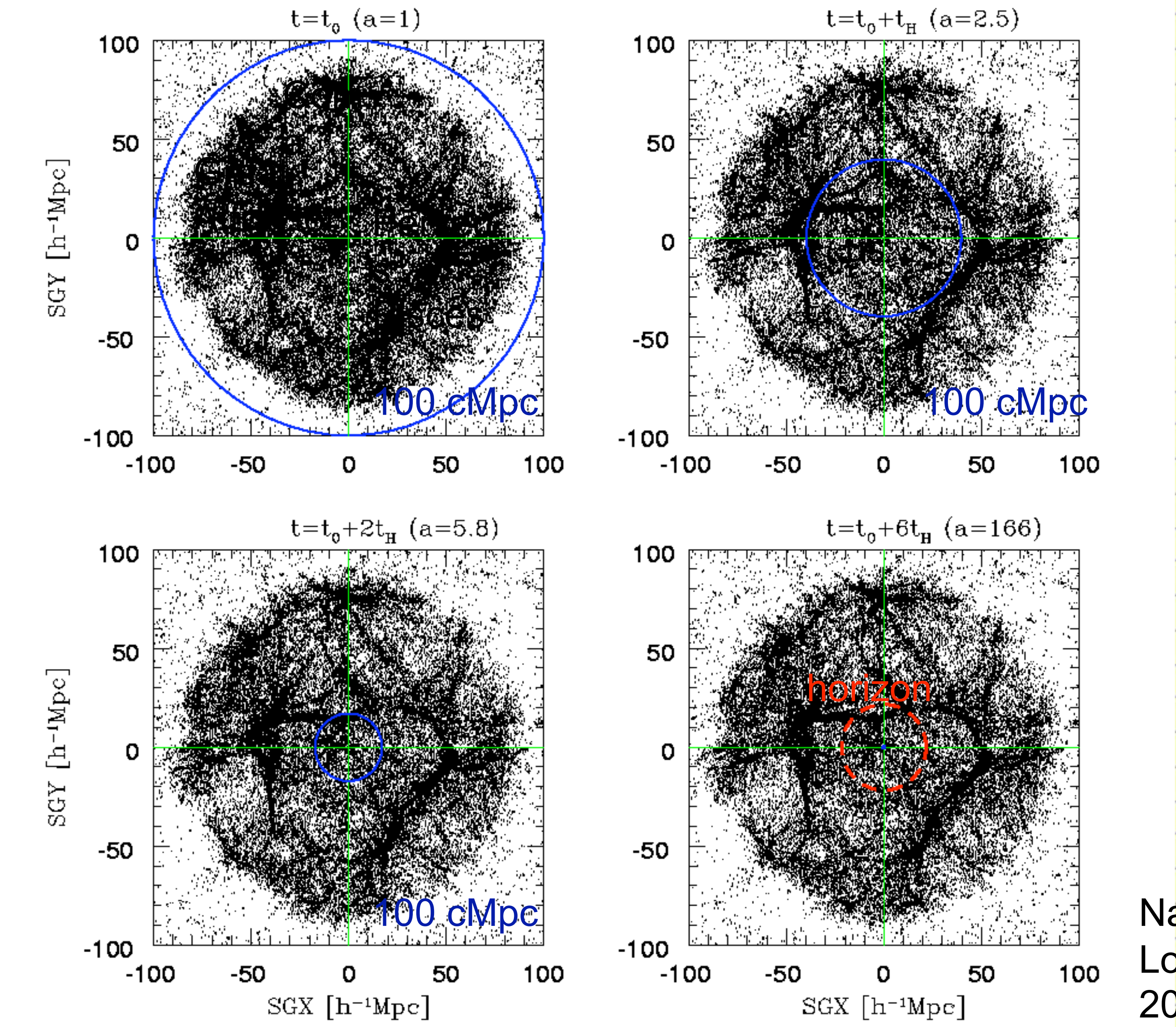
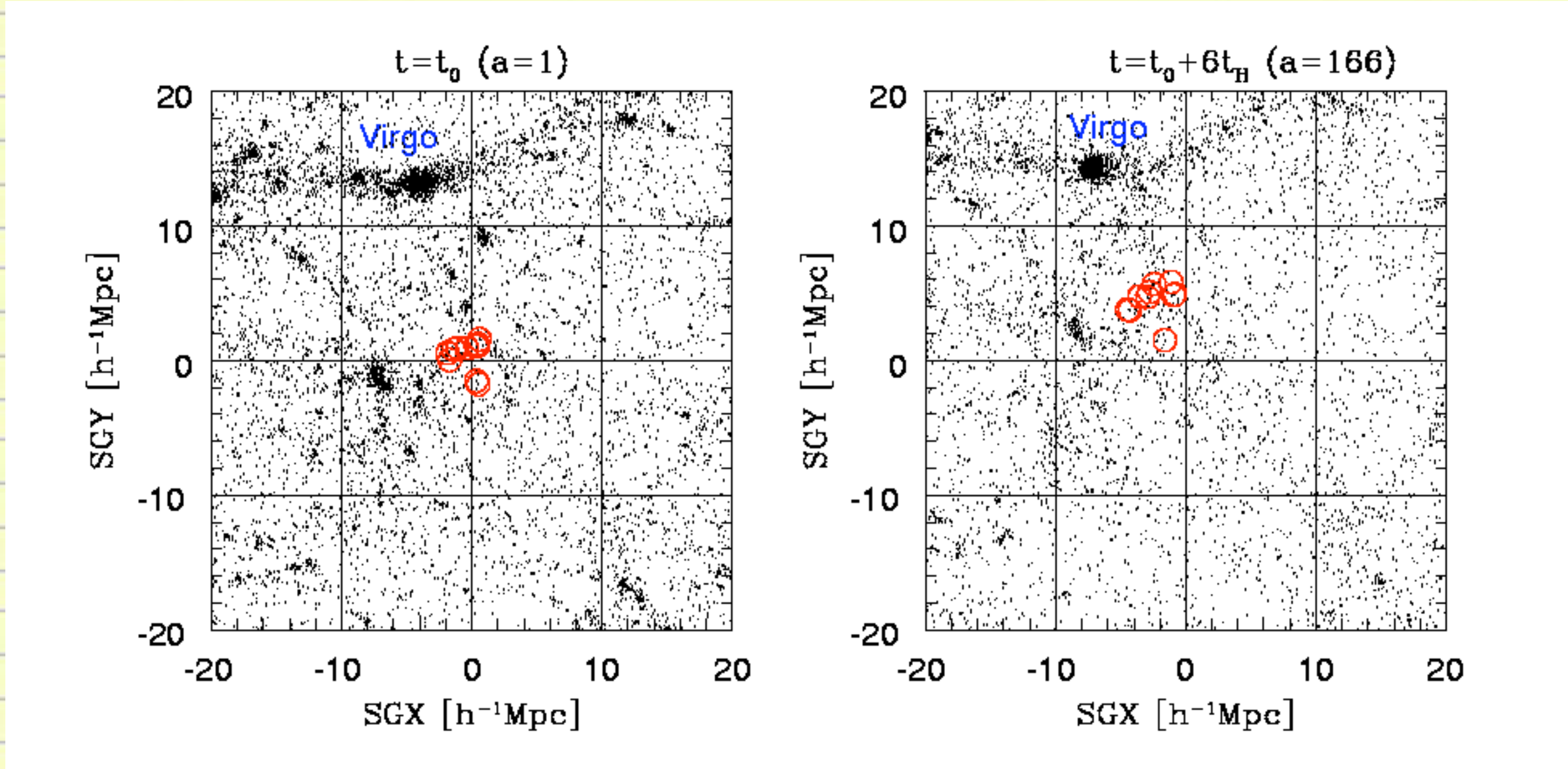


Future Evolution of Nearby Large-Scale Structure



Nagamine & Loeb 2002,2003

How many galaxies will reside within our event horizon in 100 billion years?

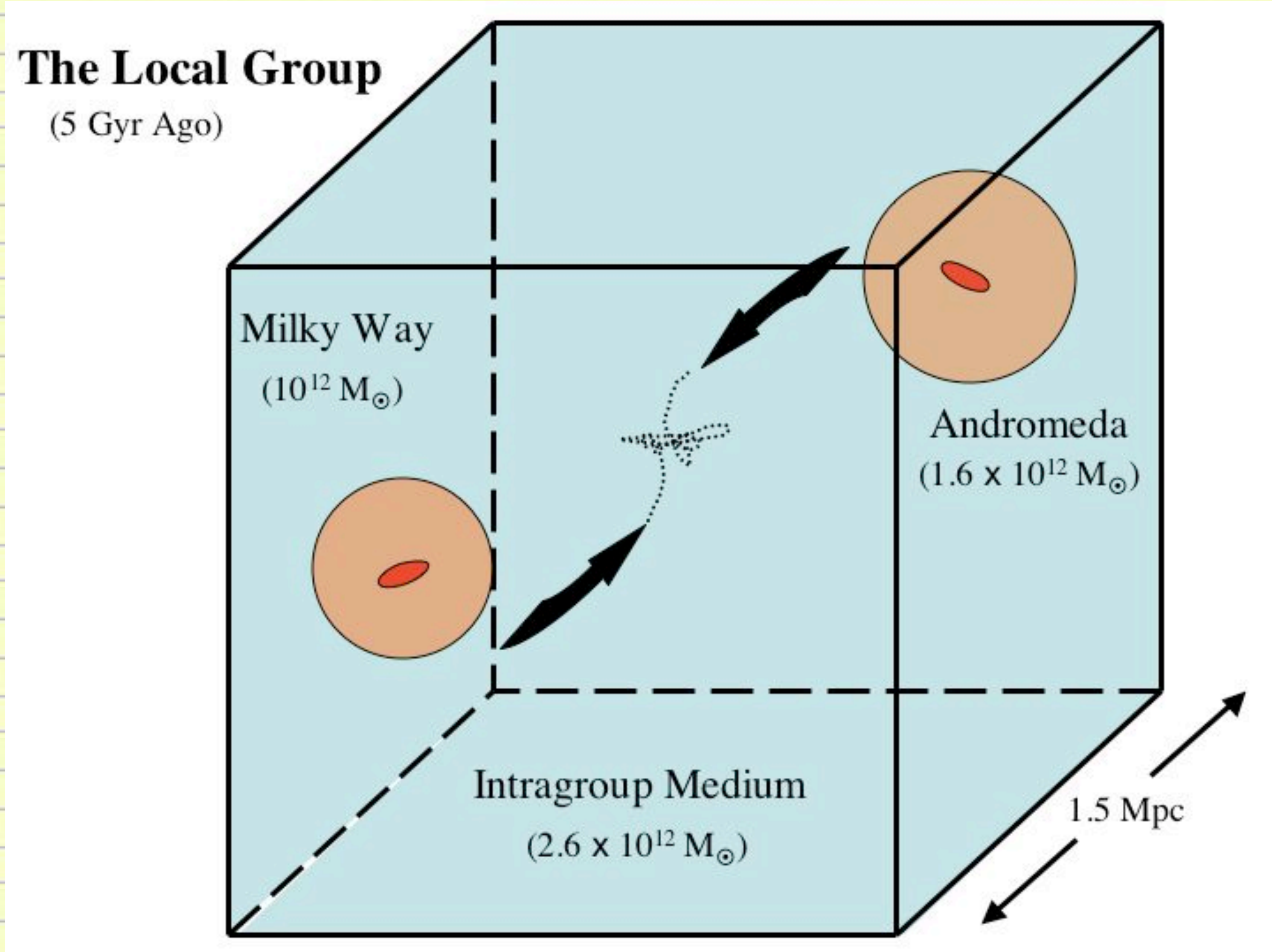


<u>Answer:</u> one surrounded by vacuum

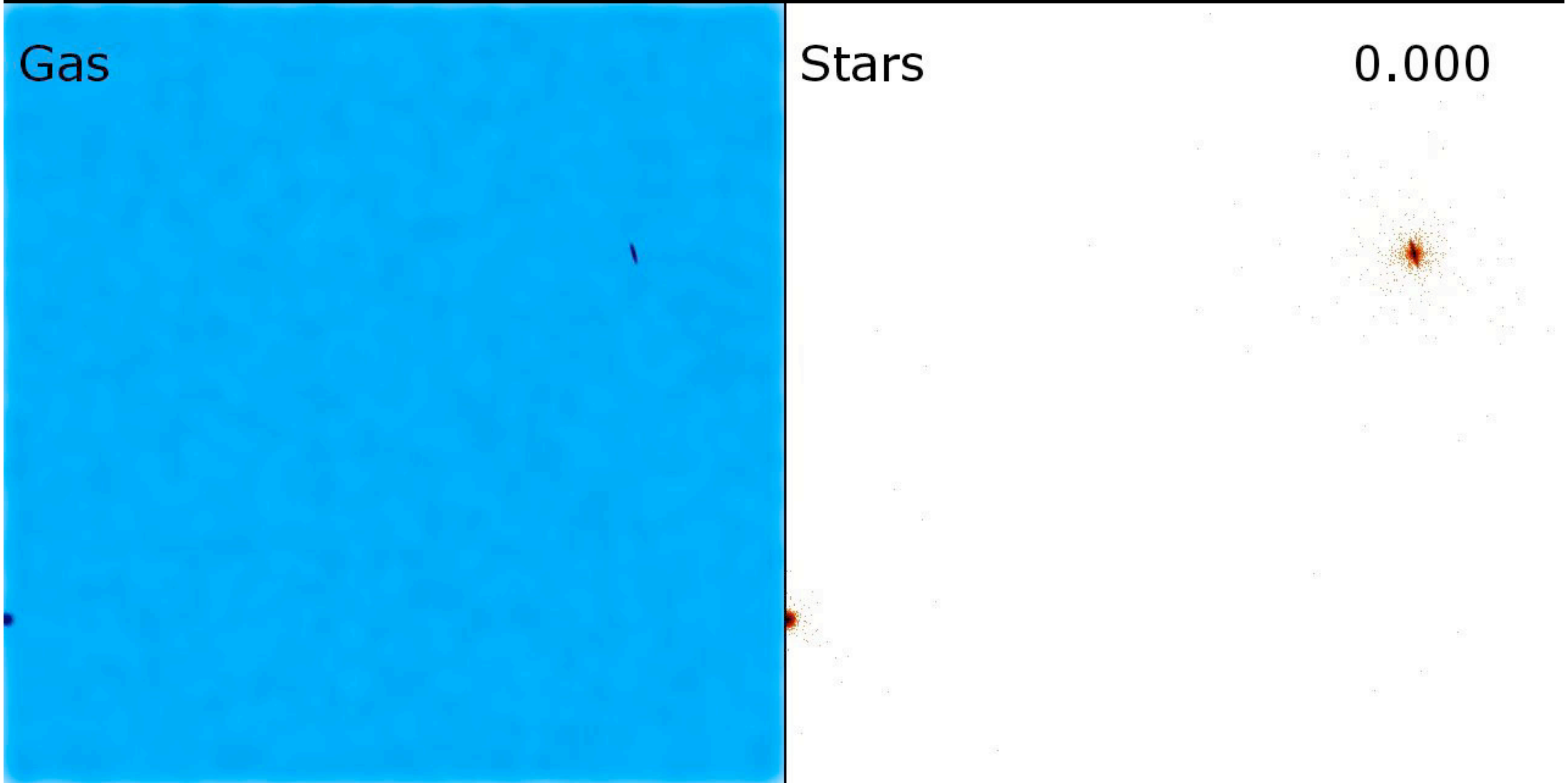
The merger product of the Andromeda and Milky-Way

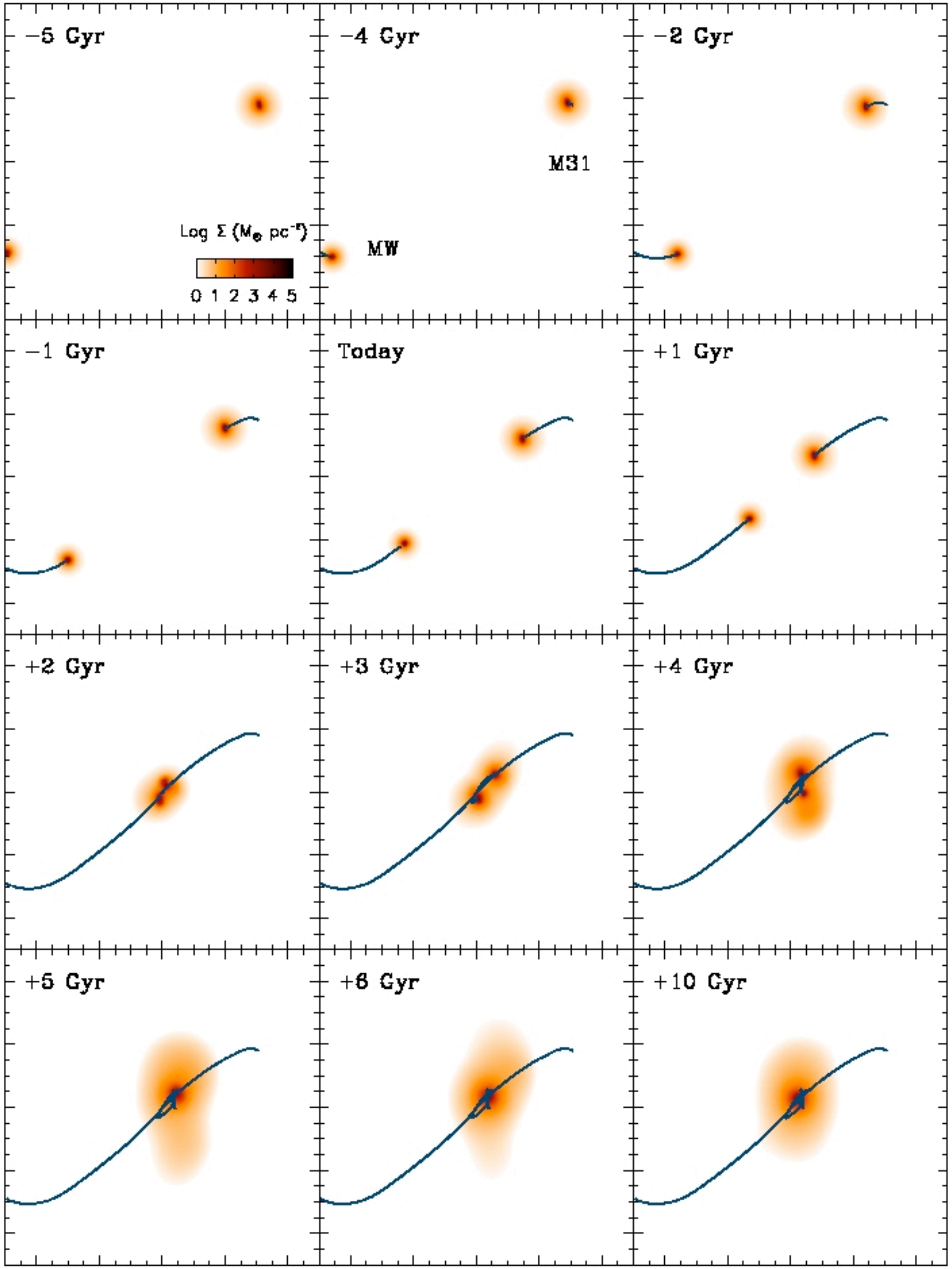
The Forthcoming Merger Between the Milky-Way and Andromeda: Milkomeda

- The merger product is the only cosmological object that will be observable to future astronomers in 100 billion years
- Collision will occur during the lifetime of the Sun
- The night sky will change
- Simulated with an N-body/hydrodynamic code (Cox & Loeb 2007)
- ***The only paper of mine that has a chance of being cited in five billion years…***


The Local Group
(5 Gyr Ago)
Milky Way
(10^{12} $M_\odot$)
Andromeda
(1.6×10^{12} $M_\odot$)
Intragroup Medium
(2.6×10^{12} $M_\odot$)
1.5 Mpc

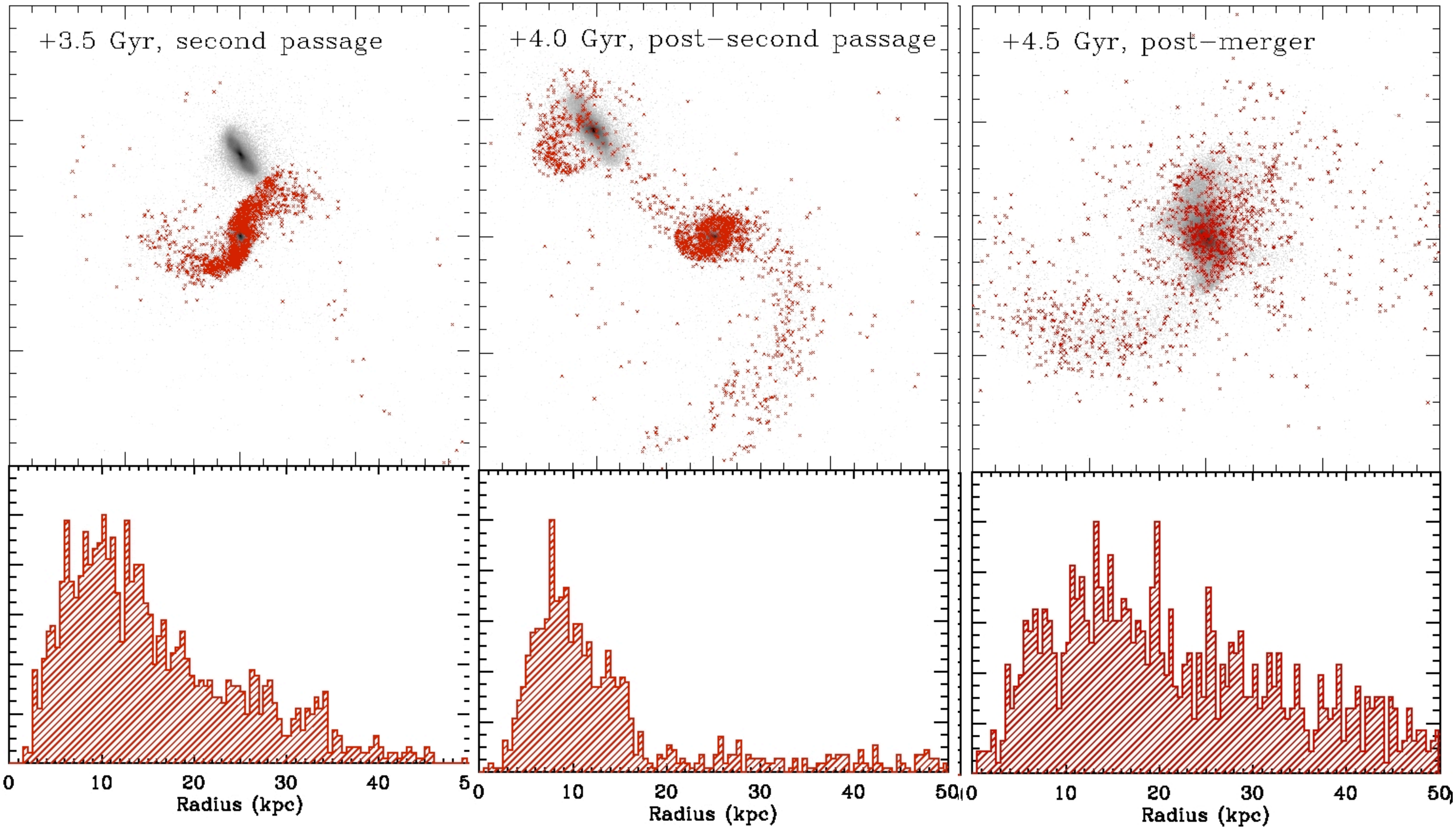
The Future Collision between the Milky Way and Andromeda Galaxies





−5 Gyr
Log Σ (M⊙ pc⁻²)
0 1 2 3 4 5
−4 Gyr
M31
MW
−2 Gyr
−1 Gyr
Today
+1 Gyr
+2 Gyr
+3 Gyr
+4 Gyr
+5 Gyr
+6 Gyr
+10 Gyr

Fate of the Sun



Highlights

- ***Cosmologists are currently exploring the scientific version of the story of genesis ("let there be light"). Future observations will utilize large-aperture infrared telescopes (for imaging the first galaxies) and low-frequency radio arrays (for imaging cosmic hydrogen in between the galaxies).***

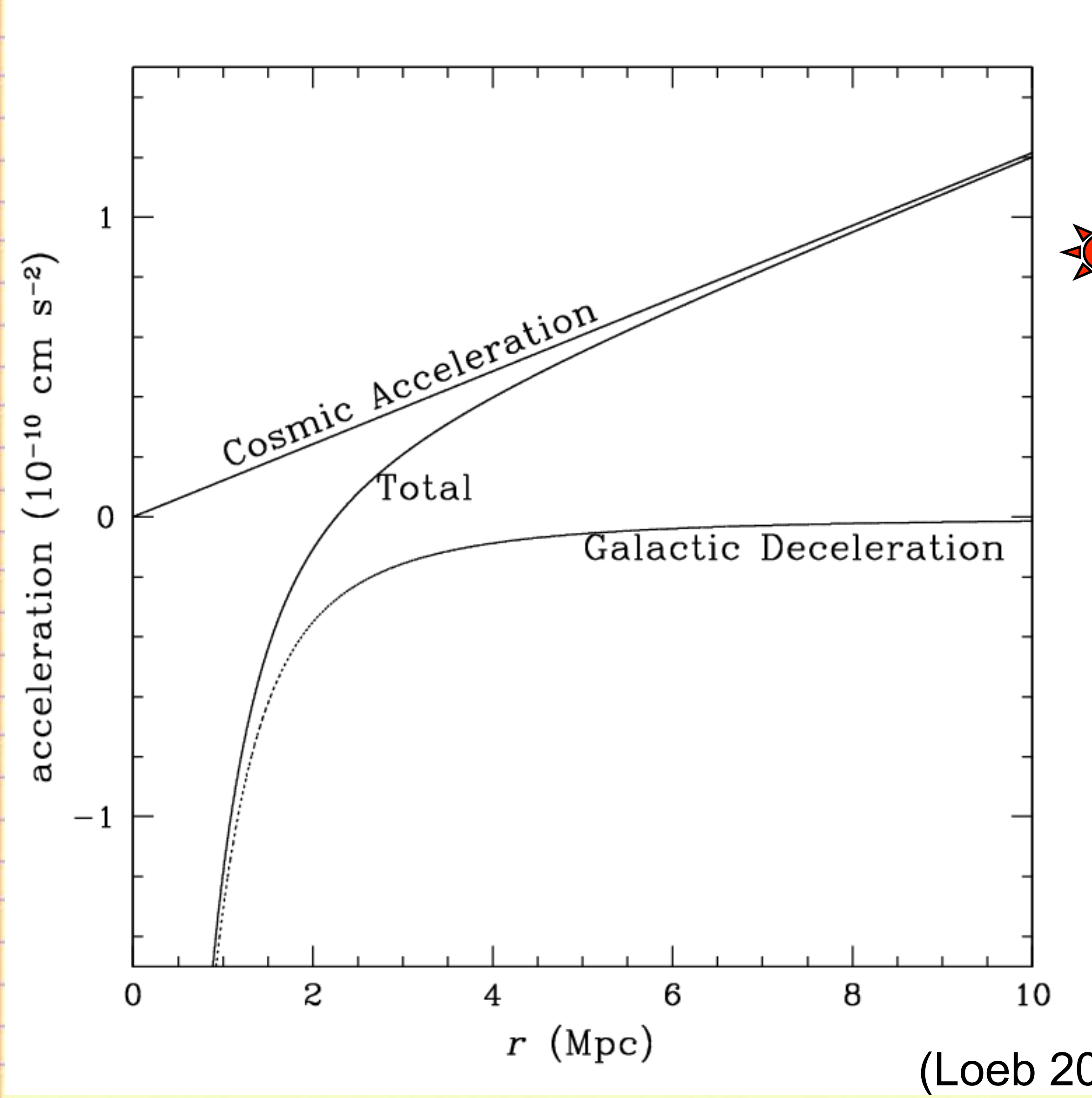
  Religious ideas about genesis are modified by science.

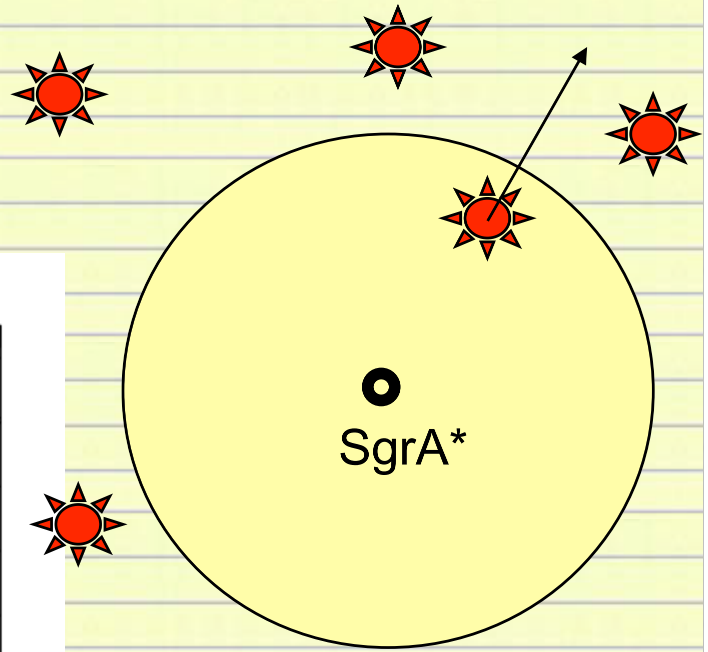
- ***The merger product of the Milky-Way and Andromeda (Milkomeda) is the only galaxy that will remain visible to us as the Universe ages by a factor of ten (a hundred billion years from now). Subsequent generations of observers will not be able to find direct evidence for the big bang.***

  Will cosmology turn into religion at that time?

- Once the Universe ages by ~100 (a trillion years from now), the wavelength of the microwave background will exceed the scale of our horizon…
- At that time, all extragalactic atoms will be pushed out of the horizon and be unavailable for tracing the cosmic expansion…

But our galaxy ejects hypervelocity stars!





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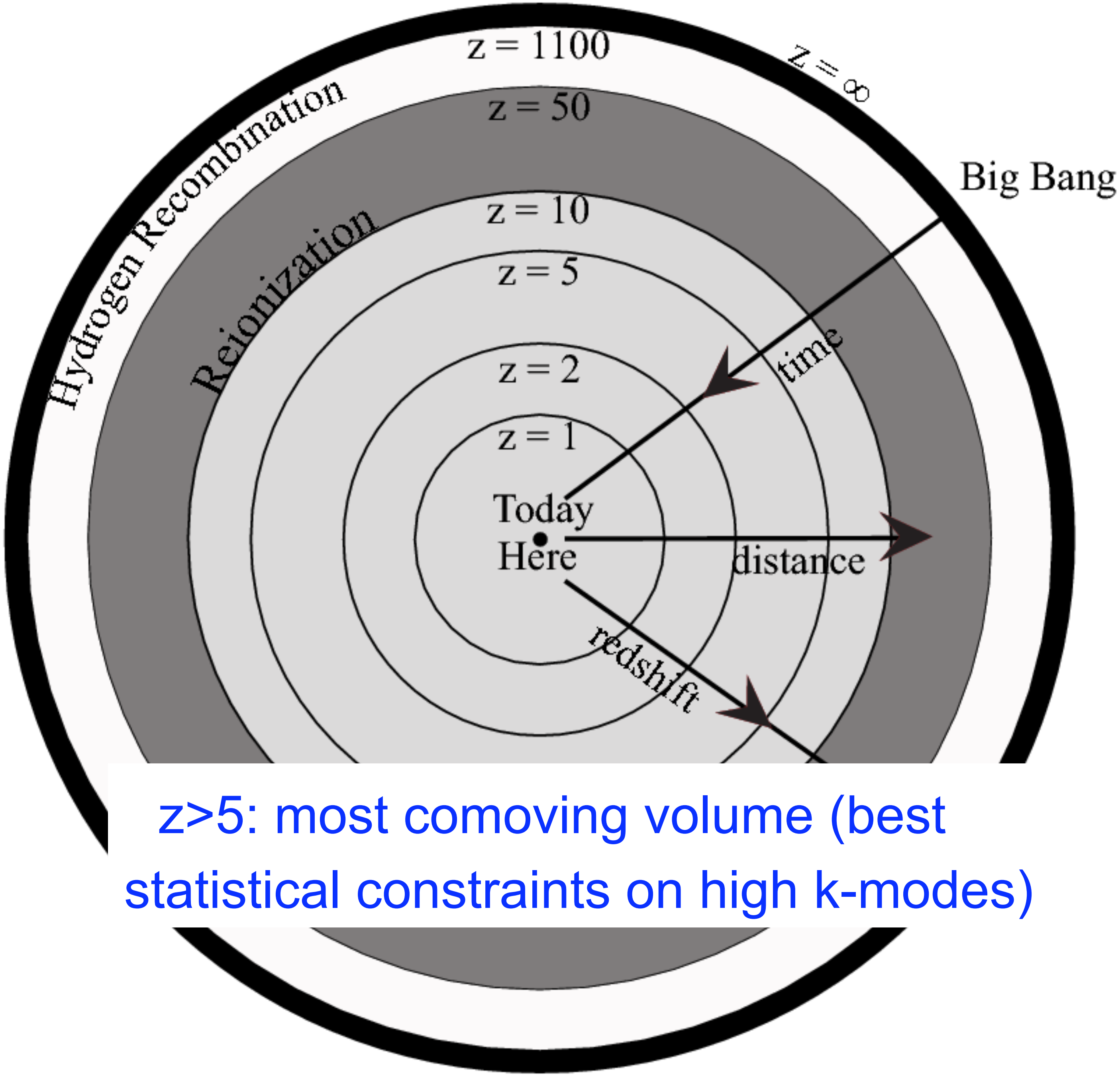
(Loeb 2011, arXiv:1102.0007)

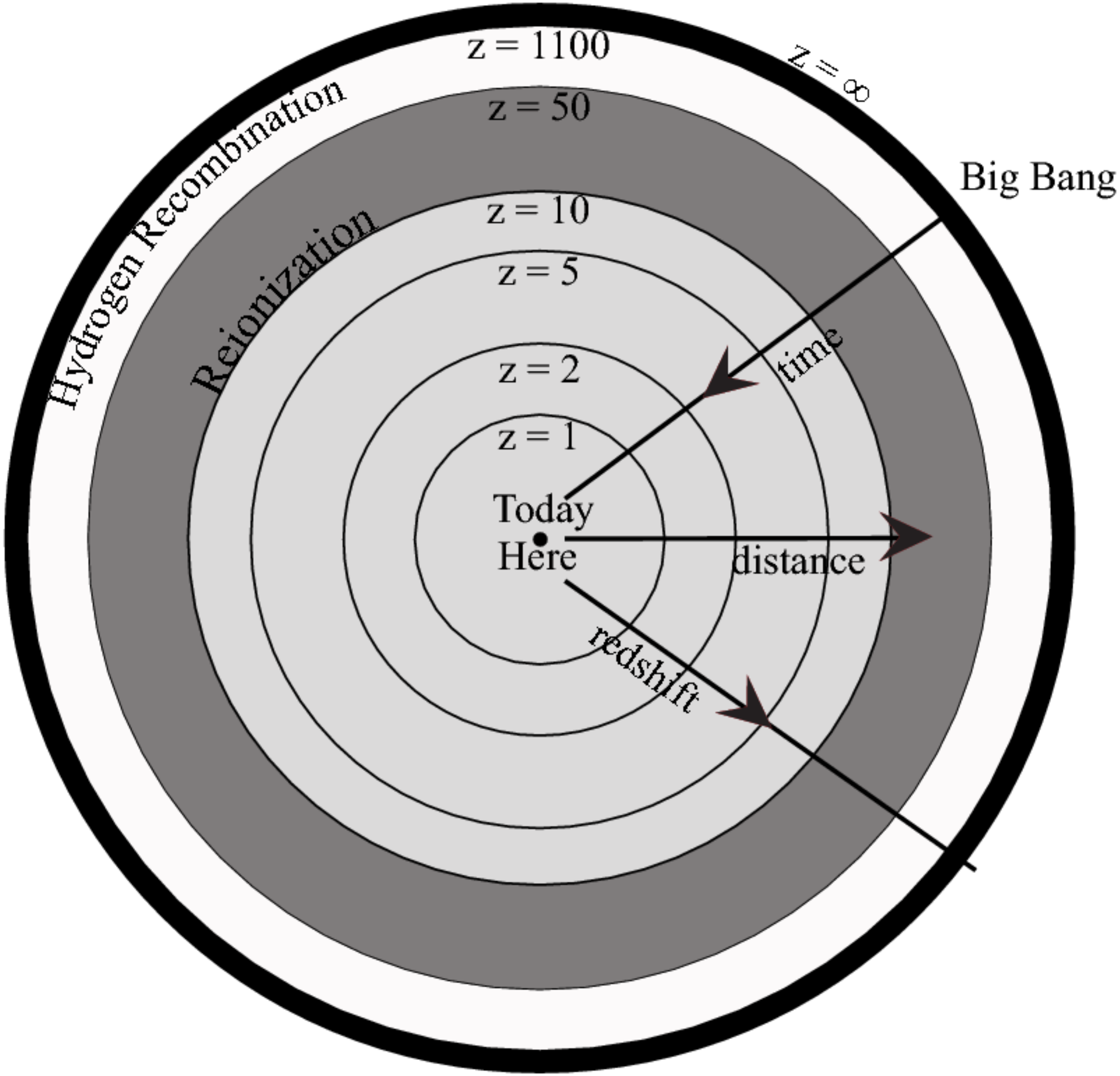
The Visible Universe



z>5: most comoving volume (best statistical constraints on high k-modes)

The Universe (14th Century, Monumental Cemetery, Pisa)

The Visible Universe



It is convenient to analyze the density perturbations in Fourier space with $\delta_{\mathbf{k}} = \int d^3r\delta(\mathbf{r})\exp\{i\mathbf{k}\cdot\mathbf{r}\}$, where $k = 2\pi/\lambda$ is the comoving wavenumber. The fractional uncertainty in the power spectrum of primordial density perturbations $P(k) \equiv \langle|\delta_{\mathbf{k}}|^2\rangle$ is given by [6, 7],

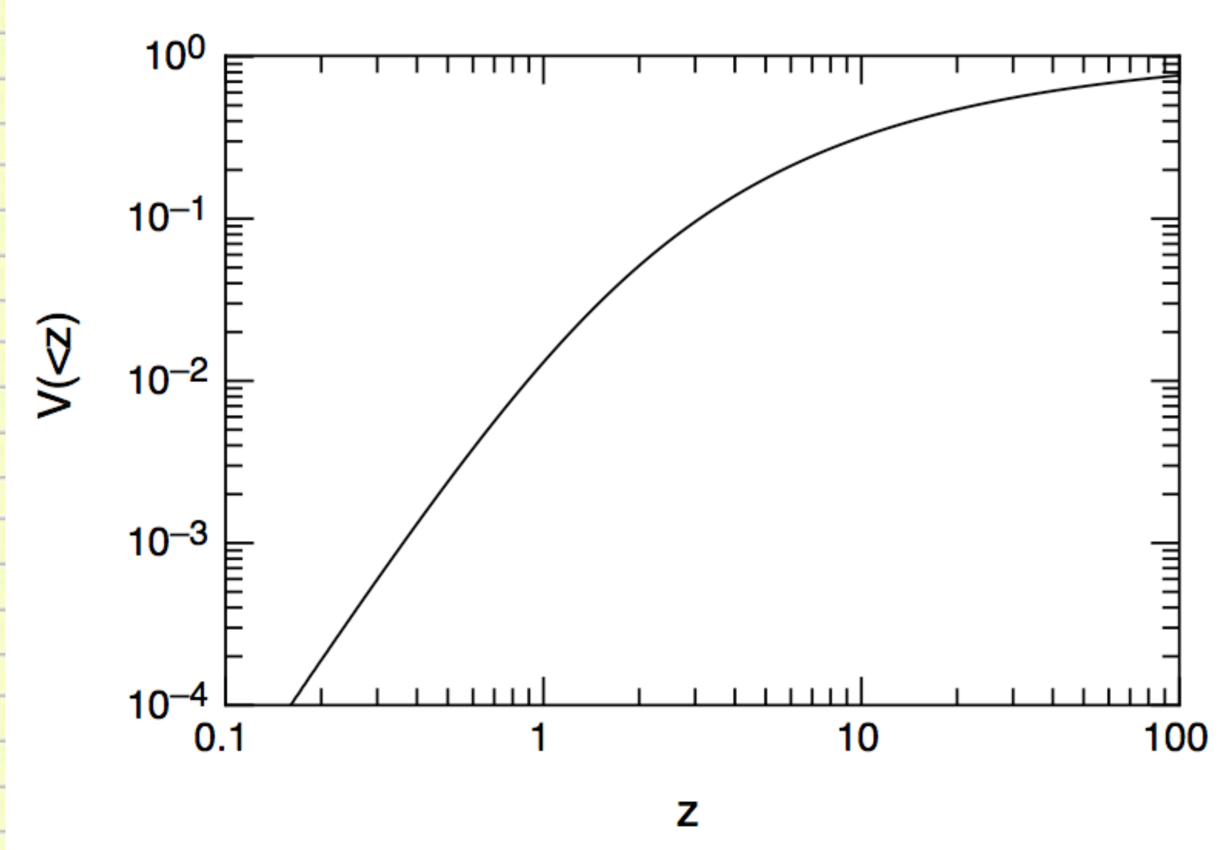
$$\frac{\Delta P(\bar{k})}{P(\bar{k})} = \frac{1}{\sqrt{N(\bar{k})}}, \qquad (1)$$

where the number of independent samples of Fourier modes with wavenumbers between k and $k + dk$ in a spherical comoving survey volume V is,

$$dN(k) = (2\pi)^{-2}k^2V dk, \qquad (2)$$

with $N(\bar{k})$ being the integral of $dN(k)/dk$ over the band of wavenumbers of interest around $\bar{k}$.

Cumulative Fraction of Comoving Volume



Loeb & Wyithe
PRL (2008)

Number of Modes $dN_{\rm 3D} = 2\pi k^2 dk \left[\mathcal{V}/(2\pi)^3\right]$

$k \equiv 2\pi/\lambda$ $dN_{\rm CMB} = \pi k dk \left[\mathcal{A}/(2\pi)^2\right]$

The Optimal Cosmic Time for Constraining the Initial Density Perturbations

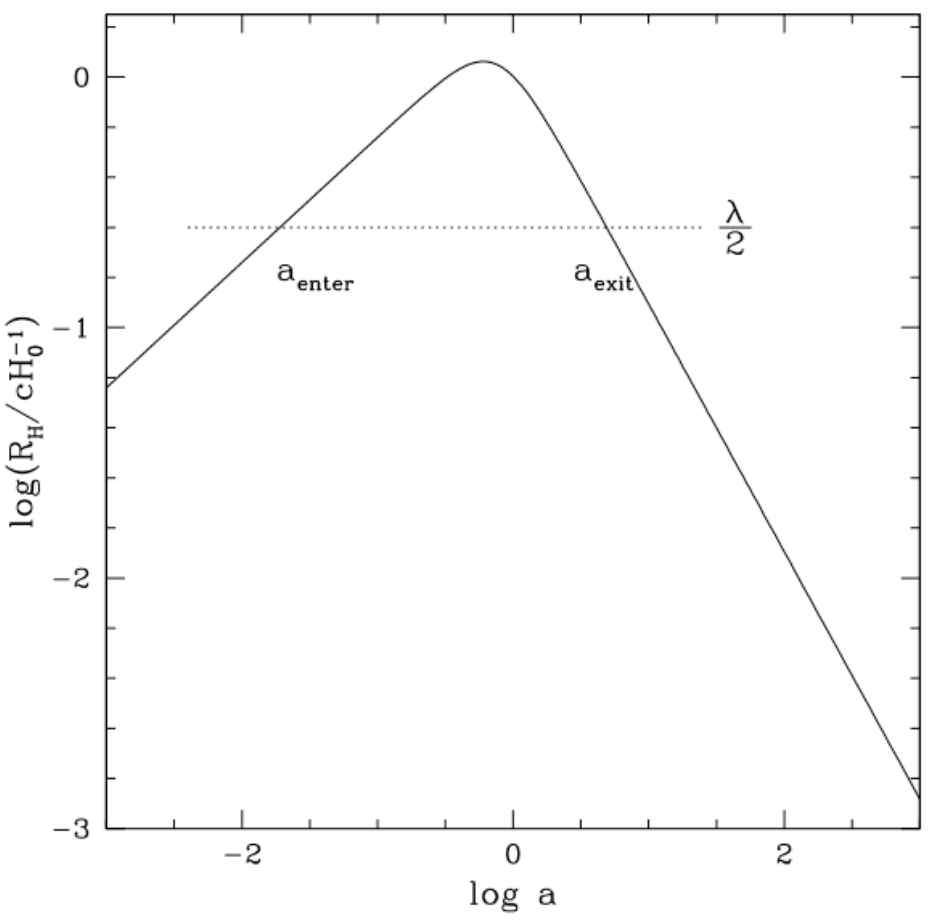


FIG. 1: In the standard (post-inflation) cosmological model, a Fourier mode with a comoving wavelength λ which enters the comoving scale of the Hubble radius $R_{\rm H} = c(aH)^{-1}$ (in units of $cH_0^{-1} = 4.3$ Gpc) at some early time (corresponding to a redshift $z = a_{\rm enter}^{-1} - 1$), would eventually exit the Hubble radius at a later time (corresponding to $a_{\rm exit}$). Hence, there is only a limited period of time when the mode can be observed.

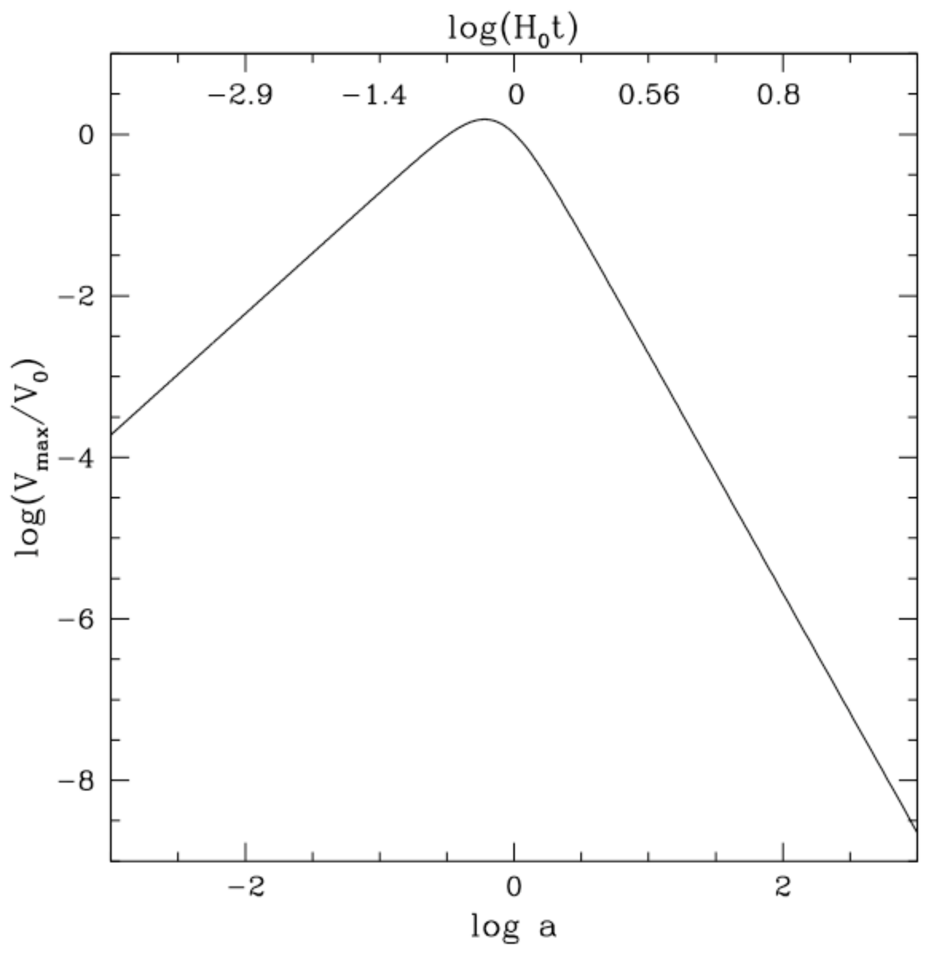
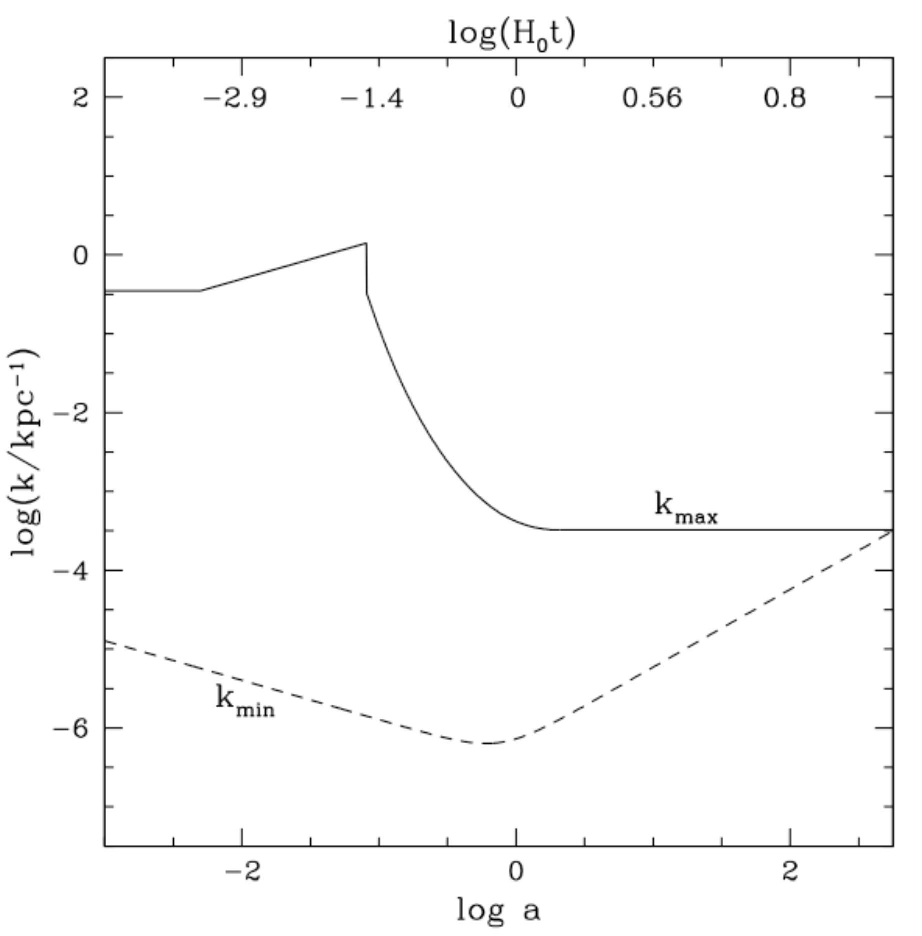
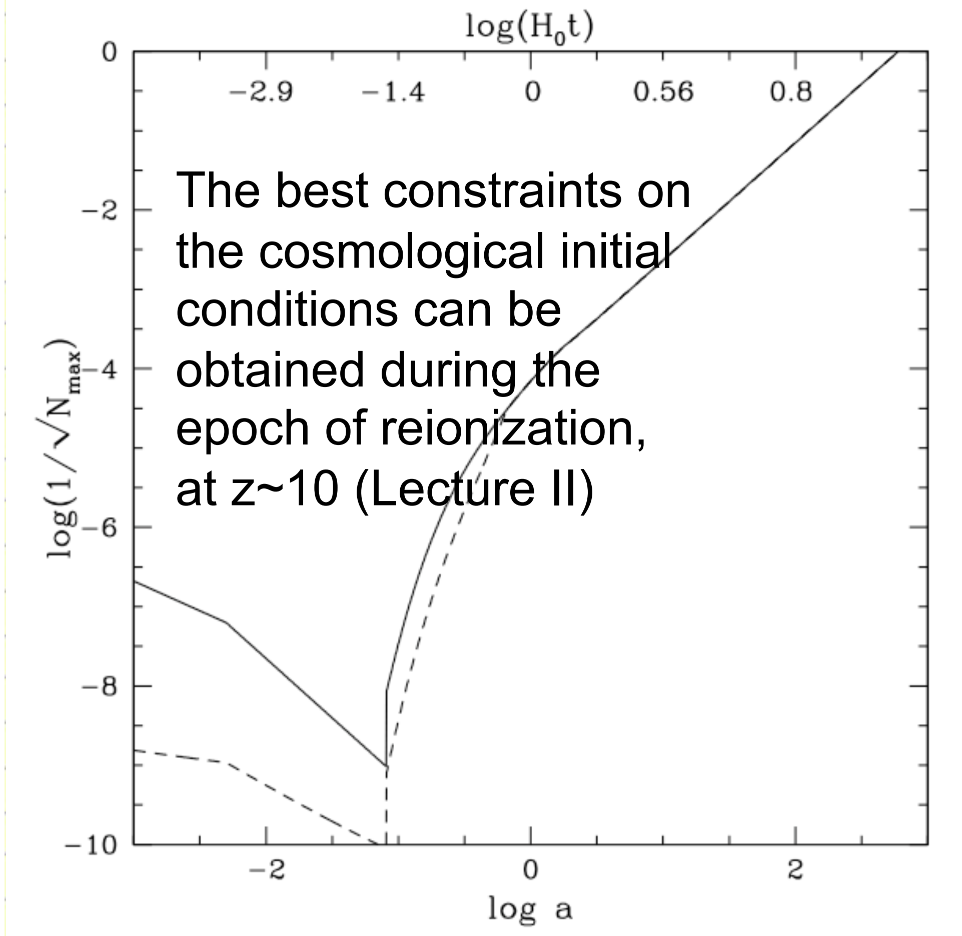


FIG. 2: The Hubble volume $V_{\rm max}$, normalized by its present-day value $V_0 = \frac{4\pi}{3}(c/H_0)^3 = 3.3 \times 10^2$ Gpc^3, as a function of cosmic time t (top axis, in units of the Hubble time $H_0^{-1} = 14$ Gyr) and scale factor $a = (1+z)^{-1}$ (bottom axis).

The Optimal Cosmic Epoch for Precision Cosmology





The best constraints on the cosmological initial conditions can be obtained during the epoch of reionization, at z~10 (Lecture II)

FIG. 3: The range of comoving wavenumbers for which the linear power spectrum can be observed per Hubble volume as a function of cosmic time and scale factor. The minimum wavelength $\lambda_{\rm min} = 2\pi/k_{\rm max}$ is taken as the larger among the baryonic Jeans scale and the scale where nonlinear structure forms at any given redshift. The maximum wavelength $\lambda_{\rm max} = 2\pi/k_{\rm min}$ is set by the Hubble diameter $2R_{\rm H}$.

FIG. 4: The minimum fractional error attainable for the power-spectrum amplitude $1/\sqrt{N_{\rm max}}$ per Hubble volume, as a function of cosmic time and scale factor (solid line). The dashed line includes the reduction in the statistical uncertainty for a present-day observer who surveys a spherical shell of comoving width $2R_H(t)$ centered at the corresponding cosmic times.